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ON RUNGE-KUTTA-NYSTROM METHODS FOR SECOND ORDER EQUATIONS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled ON RUNGE-KUTTA-NYSTROM METHODS FOR SECOND ORDER EQUATIONS submitted by David G. Barry in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis reviews Runge-Kutta-Nystrom (RKN) methods for the solution of second order ODE's. Presented are new derivations of the constraints for methods of rank three and four together with the corresponding truncation error bounds.

Two procedures are employed to obtain new coefficients for RKN methods. The use of these coefficients is evaluated with regards to truncation error bounds, accuracy and stability. Numerical experiments were performed on an IBM 360/67 and comparisons are made between the results of solving a second order equation directly by use of RKN methods and by an RK method applied to the equivalent system of first order equations. It is shown that RKN methods are in general superior to RK methods.

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PREFACE

This thesis reviews Runge-Kutta type methods for solution of second order ordinary differential equations. The solution of such an ODE by a simultaneous system of first order equations, and directly, by the Runge-Kutta-Nystrom (RKN) method, will be discussed.

New sets of coefficients for RKN methods will be presented. The solution of a second order equation using these methods will be compared with the solution by a system of equations with regards to accuracy, stability and truncation error.

CHAPTER I

INTRODUCTION

1.1 The Runge-Kutta Method

The numerical solution of differential equations is an interesting and important problem in the area of applied mathematics. In particular ordinary differential equations (ODE) of first, second and higher order are often encountered, a large class of which, though linear and of a simple form, cannot be solved analytically. This has prompted the development of several numerical methods of which the basic idea for one was first proposed by Runge [20], applied to first order equations in more accurate form by Heun [12] and Kutta [14], and is now formally called the Runge-Kutta (RK) method.

1.1.1 Basic Theory To solve numerically the initial value problem

$$(1.1) \qquad y' = f(x,y); \qquad y(x_o) = y_o$$

at a series of points $x_1, x_2 \ldots .$ Runge proposed the following formula:

$$(1.2) \qquad y_{n+1} - y_n = \sum_{i=1}^{m} \omega_i k_i$$

where

$$(1.3) \qquad k_i = hf(x_n+\alpha_i h,\ y_n+\sum_{j=1}^{i-1} \beta_{ij}k_j)$$

with $h = x_{n+1} - x_n$, $\alpha_1 = 0$ and $\beta_{ij} = 0;\ j \geq i$. The coefficients ω_i, α_i and β_{ij} are chosen such that the Taylor series expansion about (x_n,y_n) of the $\omega_i k_i$ on the right side of (1.2) matches in powers of h the expansion of y_{n+1} about x_n; i.e. the left side of (1.2),

$$(1.4) \qquad y_{n+1} - y_n = hy_n' + \frac{h^2}{2!}\, y_n'' + \frac{h^3}{3!}\, y_n^{(3)} + \frac{h^4}{4!}\, y_n^{(4)} + \ldots$$

through terms in h^{m*} for some $m* \leq m$ ($m*=m$ for $m\leq 4$) [21]. In further discussion $m*$ will be referred to as the degree of the method, m will be the rank of the method while the term order will refer to the highest ordered derivative in the ODE.

Using (1.4) up to and including terms in h^{m*+1} and the differential operator

$$(1.5) \qquad D^i = \left(\frac{\delta}{\delta x} + f\frac{\delta}{\delta y}\right)^i$$

where $f = f(x_n,y_n)$ and $\frac{\delta}{\delta x}$, $\frac{\delta}{\delta y}$ are treated as algebraic variables, an expansion for the left side of (1.2) in terms of $f(x,y)$ and its partial derivatives may be found. To

obtain the expansion for the right side of (1.2) the Taylor series for two variables, given by

$$f(x_n+h_1, y_n+h_2) = \sum_{i=0}^{\infty} \frac{1}{i!} B^i f \tag{1.6}$$

where B is the operator

$$B^i = (h_1 \frac{\delta}{\delta x} + h_2 \frac{\delta}{\delta y})^i , \tag{1.7}$$

may be used. In this case the exponent has the meaning of repeated application of B. To match powers of h through h^{m*} it is necessary to impose the restrictions

$$\alpha_i = \sum_{j=1}^{i-1} \beta_{ij}, \qquad i=2,\ldots,m. \tag{1.8}$$

1.1.2 <u>RK Methods of Rank Four</u> An example of the above procedure may be found in Ralston [18] for methods of rank two, three and four. After matching terms when $m = m^* = 4$ one obtains the constraints given in matrix form by (1.9).

$$
(1.9)\qquad
\begin{array}{l}
hf \\[6pt]
\frac{h^2}{2!}\, Df \\[6pt]
\frac{h^3}{3!}\, D^2 f \\[6pt]
\frac{h^3}{3!}\, f_y Df \\[6pt]
\frac{h^4}{4!}\, D^2 f \\[6pt]
\frac{h^4}{4!}\, f_y D^2 f \\[6pt]
\frac{h^4}{4!}\, DfDf_y \\[6pt]
\frac{h^4}{4!}\, f_y^2 Df
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2\alpha_2 & 2\alpha_3 & 2\alpha_4 \\
0 & 3\alpha_2^2 & 3\alpha_3^2 & 3\alpha_4^2 \\
0 & 0 & 6\alpha_2\beta_{32} & 6(\alpha_2\beta_{42}+\alpha_3\beta_{43}) \\
0 & 4\alpha_2^3 & 4\alpha_3^3 & 4\alpha_4^3 \\
0 & 0 & 12\alpha_2^2\beta_{32} & 12(\alpha_2^2\beta_{42}+\alpha_3^2\beta_{43}) \\
0 & 0 & 8\alpha_2\alpha_3\beta_{32} & 8\alpha_4(\alpha_2\beta_{42}+\alpha_3\beta_{43}) \\
0 & 0 & 0 & 24\alpha_2\beta_{32}\beta_{43}
\end{bmatrix}
\cdot
\begin{bmatrix}
\omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1
\end{bmatrix}
$$

These constraints and the three restrictions imposed by (1.8) form a system of eleven non-linear equations in thirteen unknowns. This two parameter system can be solved in terms of α_2 and α_3 with the obvious requirement that the denominator not be zero for any of the expressions giving the remaining coefficients. Several methods of degree four have been derived as above simply by choosing values of α_2 and α_3 [19].

1.1.2.1 Truncation Error By maintaining and matching terms in h^5 Lotkin [15] arrived at a bound, independent of $f(x,y)$, on the most significant part of the truncation error involved in an RK method of degree four. This was accomplished by assuming the following in a region R about (x_n,y_n) containing all points in (1.3):

$$(1.10) \qquad |f(x,y)| < M; \qquad \left|\frac{\delta^{i+j} f}{\delta x^i \delta y^j}\right| < \frac{L^{i+j}}{M^{j-1}}$$

where $i + j \leq 4$, and L and M are positive constants independent of x and y. The truncation error may then be written as

$$(1.11) \qquad T_4 = h^5 t_4 + O(h^6)$$

where $|t_4|$ is given by

$$(1.12)\qquad |t_4| < [16|u_1| + 4|u_2| + |u_2+3u_3| + |2u_2+3u_3| + |u_2+u_3| + |u_3| + 8|u_4| + |u_5| + |2u_5+u_7| + |u_5+u_6+u_7| + |u_6| + |2u_6+u_7| + |u_7| + 2|u_8|]ML^4$$

and

$$(1.13)\qquad u_1 = 1/120 - (\alpha_2^4\omega_2+\alpha_3^4\omega_3+\alpha_4^4\omega_4)/24 \ ,$$

$$(1.14)\qquad u_2 = 1/20 - [\alpha_2\alpha_3^2\beta_{32}\omega_3 + \alpha_4^2(\alpha_2\beta_{42}+\alpha_3\beta_{43})\omega_4]/2 \ ,$$

$$(1.15)\qquad u_3 = 1/120 - [\alpha_2^3\beta_{32}\omega_3 + (\alpha_2^3\beta_{42}+\alpha_3^3\beta_{43})\omega_4]/6 \ ,$$

$$(1.16)\qquad u_4 = 1/30 - [\alpha_2^2\alpha_3\beta_{32}\omega_3 + \alpha_4(\alpha_2^2\beta_{42}+\alpha_3^2\beta_{43})\omega_4]/2 \ ,$$

$$(1.17)\qquad u_5 = 1/120 - \alpha_2^2\beta_{32}\beta_{43}\omega_4 \ ,$$

$$(1.18)\qquad u_6 = 1/40 - [\alpha_2^2\beta_{32}^2\omega_3 + (\alpha_2\beta_{42}+\alpha_3\beta_{43})^2\omega_4]/2 \ ,$$

$$(1.19) \qquad u_7 = 7/120 - \alpha_2(\alpha_3+\alpha_4)\beta_{32}\beta_{43}\omega_4 \ ,$$

$$(1.20) \qquad u_8 = 1/120 \ .$$

Ralston [18] proceeded further, obtaining coefficients for methods of degree two, three and four which minimized this type of bound.

1.1.3 Advantages and Disadvantages Ralston [19] cites the advantages and disadvantages of RK methods. The three most advantageous properties are:

1. They are single step methods, i.e. to find y_{n+1} it is necessary only to have the preceeding values of y_n and x_n, and the step-size h.
2. They are self-starting, the interval between steps may be changed at will and they are particularly easy to apply on a digital computer.
3. They are comparable in accuracy to predictor corrector methods of the same degree.

The drawbacks to these methods are twofold:

1. The step-size generally must be chosen conservatively because of the difficulty in obtaining the per-step error, i.e. h is smaller than is actually necessary in order to achieve the

desired accuracy.

2. They require a number of evaluations of $f(x,y)$ at each step equal to or greater than the degree of the method, whereas the corresponding predictor-corrector method requires a number of function evaluations less than the degree.

It is because of the above restrictions that RK methods are in general used only to obtain starting values for predictor-corrector methods and in changing the step-size.

1.2 Summary of Literature

Work has continued on RK methods for first, second and higher order equations. Bieberbach [1], as well as Lotkin, did extensive research on the truncation error involved in these methods while Blum [2], Ceschino and Kuntzmann [3], and Gill [10] presented new sets of coefficients having specific attributes. Nystrom [16] developed methods of degree greater than four and Frey [7], who refers to these higher degree methods as Runge-Kutta-Nystrom methods, improved upon them by choosing coefficients such that the number of function evaluations necessary for each step is less than the rank of the method for $m \geq 5$.

Nystrom [16] also extended RK methods to second order equations and Zurmuhl continued this extension, first to third order equations [22], and then to equations of the n-th order [23]. Nystrom restricted his work for second order equations to methods of rank three and four while Zurmuhl concerned himself only with methods of rank four and function evaluations at x_n, $x_n + h/2$ and $x_n + h$. Nystrom [16] provides a set of coefficients for second order equations while Zurmuhl [23] presents computing schemes for n-th order equations where $3 \le n \le 10$.

Froese [9] evaluated the RK type methods suggested by Zurmuhl using Rademacher's theory of error propagation [17]. Her work was primarily concerned with the fourth order initial value problem

$$(1.21) \qquad y^{(4)} = y; \quad y^{(i)}(x_0) = y_0^{(i)}; \quad i=0,\ldots,3,$$

and the question of whether it should be solved as a system of four first order equations, two second order equations or by use of a direct method for fourth order equations. Zurmuhl [23] states that solving an n-th order equation directly is superior to solving an equivalent system of equations if $n \ge 3$. This conclusion was based on a consideration that the truncation error for

the RK method of rank four is of degree $(n+2)$ for $n \geq 2$. Froese's analysis and experimental results confirmed Zurmuhl's conclusion.

1.3 Purpose of Study

Any direct method for second order equations will hereafter be called a Runge-Kutta-Nystrom (RKN) method. The purpose of the present investigation is to present a new derivation of the general RKN method of rank m and degree m^*, and to obtain several new sets of coefficients for methods of rank four. The coefficient constraints for RKN methods of rank three and four are derived with a new restriction reducing the number of equations in these systems. Bounds for the truncation error of these methods are also found using the approach of Lotkin [15].

Several second order ODE's are solved using a system of two first order equations and the corresponding RKN methods. Comparing these results and the bounds on the truncation error involved in both methods it is shown that Zurmuhl's result [23] could be extended to include $n = 2$.

CHAPTER II

RUNGE-KUTTA METHODS FOR SOLUTION OF SECOND ORDER ODE's

2.1 As a System of First Order Equations

An r-th order ODE

$$(2.1) \qquad y^{(r)} = f(x,y,y',\ldots,y^{(r-1)})$$

$$y^{(i)}(x_0) = y_0^{(i)}; \qquad i=0,1,\ldots,r-1$$

may be solved as a system of lower order equations if so desired [9]. For simplicity the order of all the equations in the system may be chosen to be one. In this case (2.1) becomes

$$(2.2) \qquad Y' = F(x,Y); \qquad Y(x_0) = Y_0$$

and may be solved by applying RK methods [11]

$$(2.3) \qquad Y_{n+1} = Y_n + \sum_{i=1}^{m} \omega_i K_i$$

where

$$(2.4) \qquad K_i = hF(x_n+\alpha_i h,\ Y_n+\sum_{j=1}^{i-1} \beta_{ij} K_j) \ .$$

The variables Y, K_i and K_j denote vectors of length r, i.e. $K_i = (k_{i1}, k_{i2}, \ldots, k_{ir})$.

Hildebrand [13] considers the second order initial value problem

$$(2.5) \qquad y'' = f(x,y,y'); \quad y(x_0) = y_0; \quad y'(x_0) = y'_0 \;,$$

treating it as the system of first order equations

$$(2.6) \qquad y' = z; \quad y(x_0) = y_0$$

and

$$(2.7) \qquad z' = f(x,y,z); \quad z(x_0) = z_0 \;,$$

and suggesting the solution by means of a RK method of rank four. Extending this to a method of rank m would give the solution of (2.6-7) and consequently (2.5) in the following manner:

$$(2.8) \qquad y_{n+1} = y_n + \sum_{i=1}^{m} \omega_i k_i$$

and

$$(2.9) \qquad z_{n+1} = z_n + \sum_{i=1}^{m} \omega_i \ell_i$$

where

$$k_i = h(z_n + \sum_{j=1}^{i-1} \beta_{ij} \ell_j) \tag{2.10}$$

and

$$\ell_i = hf(x_n + \alpha_i h,\ y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j,\ z_n + \sum_{j=1}^{i-1} \beta_{ij} \ell_j)\ . \tag{2.11}$$

The above computational procedure where $m = 4$ is that most commonly used for solving (2.5) using RK methods.

2.2 Derivation of the RKN Method

The remainder of this Chapter presents a new derivation of the general RKN method. Nystrom [16] derived RKN methods of rank three and four while Zurmuhl [22], [23] worked only with methods of rank four for third and higher ordered equations. To develop the general RKN method of rank m define

$$y_n' = \frac{v_n}{h}\ ;\quad \ell_i = \frac{2!}{h} k_i^*\ . \tag{2.12}$$

Substituting (2.12) into (2.10) gives

$$k_i = v_n + 2! \sum_{j=1}^{i-1} \beta_{ij} k_j^*\ , \tag{2.13}$$

and subsequently using (2.12-13) in (2.11) yields

$$(2.14)\qquad \ell_i = \frac{2!}{h} k_i^* = hf(x_n+\alpha_i h,\ y_n+(\sum_{j=1}^{i-1}\beta_{ij})v_n$$

$$+2!\sum_{j=1}^{i-1}\beta_{ij}(\sum_{p=1}^{j-1}\beta_{jp}k_p^*),\ z_n+\frac{2!}{h}\sum_{j=1}^{i-1}\beta_{ij}k_j^*) \ .$$

From (2.13-14) (2.8-9) becomes

$$(2.15)\qquad y_{n+1} = y_n + (\sum_{i=1}^{m}\omega_i)v_n + 2!\sum_{i=1}^{m}\omega_i(\sum_{j=1}^{i-1}\beta_{ij}k_j^*)$$

and

$$(2.16)\qquad z_{n+1} = z_n + \frac{2!}{h}\sum_{i=1}^{m}\omega_i k_i^* \ .$$

Imposing the following constraints included in any RK method of rank m:

$$(2.17)\qquad \alpha_i = \sum_{j=1}^{i-1}\beta_{ij}; \qquad i=2,3,\ldots,m$$

and

$$(2.18)\qquad \sum_{i=1}^{m}\omega_i = 1 \ ,$$

and multiplying (2.16) by h gives for (2.15-16)

(2.19) $$y_{n+1} = y_n + v_n + 2! \sum_{i=1}^{m} \omega_i \left(\sum_{j=1}^{i-1} \beta_{ij} k_j^* \right)$$

and

(2.20) $$v_{n+1} = v_n + 2! \sum_{i=1}^{m} \omega_i k_i^*$$

where

(2.21) $$k_i^* = \frac{h^2}{2!} f(x_n+\alpha_i h,\ y_n+\alpha_i v_n+2! \sum_{j=1}^{i-1} \beta_{ij} \left(\sum_{p=1}^{j-1} \beta_{jp} k_p^* \right),$$

$$(v_n+2! \sum_{j=1}^{i-1} \beta_{ij} k_j^*)/h) \ .$$

Applying (2.19-21) directly to (2.5) is thus equivalent to using (2.8-11) in solving (2.6-7). The above leads to the formulation of the general RKN method which may be conveniently given by

(2.22) $$y_{n+1} = y_n + v_n + \sum_{i=1}^{m} w_{0i} k_i$$

and

(2.23) $$v_{n+1} = v_n + \sum_{i=1}^{m} w_{1i} k_i$$

where

$$k_i = \frac{h^2}{2!} f(x_n + a_i h,\ y_n + a_i v_n + \sum_{j=1}^{i-1} b_{ij} k_j, \tag{2.24}$$

$$(v_n + 2! \sum_{j=1}^{i-1} c_{ij} k_j)/h).$$

2.2.1 <u>Coefficients for an RKN Method</u> Matching corresponding terms in (2.19-21) and the general RKN method given by (2.22-24) yields coefficients for a RKN method of rank m in terms of the coefficients of any RK method of corresponding rank. These are given by

$$a_i = \alpha_i; \quad i=1,2,\ldots,m$$

$$b_{ij} = 0; \quad i=1,2,\ldots,m; \quad j \geq i-1$$

$$b_{ij} = 2! \sum_{p=j+1}^{i-1} \beta_{ip}\beta_{pj}; \quad i=3,4,\ldots,m \quad j=1,2,\ldots,i-2$$

$$c_{ij} = \beta_{ij}; \quad i=1,2,\ldots,m; \quad j=1,2,\ldots,m \tag{2.25}$$

$$w_{0m} = 0$$

$$w_{0j} = 2! \sum_{i=j+1}^{m} \omega_i \beta_{ij}; \quad j=1,2,\ldots,m-1$$

$$w_{1j} = 2!\omega_j; \quad j=1,2,\ldots,m .$$

These relationships between RK and RKN methods enable one to obtain new coefficients for RKN methods of any rank if the coefficients for an RK method of the same rank are known.

2.2.2 Coefficient Format The coefficients for RK methods of rank m will be given throughout in the following form:

$$\begin{array}{cccc|c}
\beta_{21} & & & & \alpha_2 \\
\beta_{31} & \beta_{32} & & & \alpha_3 \\
\vdots & \vdots & & & \vdots \\
\beta_{m1} & \beta_{m2} & \cdots & \beta_{m,m-1} & \alpha_m \\
\hline
\omega_1 & \omega_2 & \cdots & \omega_{m-1} & \omega_m
\end{array} \tag{2.26}$$

The corresponding RKN format will be

$$
(2.27)\qquad
\begin{array}{|cccc|c|}
\hline
b_{21} & & & & a_2^2 \\
b_{31} & b_{32} & & & a_3^2 \\
\vdots & \vdots & & & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{m,m-1} & a_m^2 \\
\hline
c_{21} & & & & a_2 \\
c_{31} & c_{32} & & & a_3 \\
\vdots & \vdots & & & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{m,m-1} & a_m \\
\hline
w_{01} & w_{02} & \cdots & w_{0,m-1} & w_{0,m} \\
w_{11} & w_{12} & \cdots & w_{1,m-1} & w_{1,m} \\
\hline
\end{array}
\quad .
$$

The reason for using this format will become clear in the following Chapter. All coefficients restricted to a value of zero are not included.

CHAPTER III

DERIVATION OF CONSTRAINTS FOR RUNGE-KUTTA-NYSTROM METHODS

3.1 Constraints for a Method of Rank Four

The classic derivation of the RK method of rank four was outlined briefly in Chapter I to indicate the manner in which the technique was originally formulated. The same pattern will be followed in more detail in building similar constraints for RKN methods of the same rank.

Consider (2.22-23) expressed in the form

$$y_{n+1} - y_n - v_n = \sum_{i=1}^{m} w_{0i} k_i \tag{3.1}$$

and

$$v_{n+1} - v_n = \sum_{i=1}^{m} w_{1i} k_i \ . \tag{3.2}$$

To derive the constraints for a RKN method of rank m it is necessary to expand the left and right sides of (3.1-2) by means of Taylor series. The coefficients of the method are chosen in such a manner that the expansions of each side of (3.1) have matching terms in powers of h^2 through h^{m*} and in (3.2) through h^{m*+1}, the

latter because of the definition for v_n given by (2.12).

3.1.1 <u>The Taylor Series Expansion</u> Expanding y_{n+1} about $x = x_n$ using the Taylor series for a function of one variable given by (1.4) and replacing hy'_n with v_n according to (2.12) produces

$$(3.3) \qquad y_{n+1} - y_n - v_n = \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y_n^{(3)} + \frac{h^4}{4!} y_n^{(4)} + \frac{h^5}{5!} y_n^{(5)} + \ldots$$

for the expansion of the left side of (3.1). Similarly the Taylor series expansion of v_{n+1} about $x = x_n$ with derivatives of v_n replaced by the derivatives of y_n from (2.12) gives

$$(3.4) \qquad v_{n+1} - v_n = h^2 y''_n + \frac{h^3}{2!} y_n^{(3)} + \frac{h^4}{3!} y_n^{(4)} + \frac{h^5}{4!} y_n^{(5)} + \ldots$$

for the left side of (3.2).

For the method of rank four expressions for $y^{(3)}$, $y^{(4)}$ and $y^{(5)}$ must be found in terms of $f(x,y,y')$. For compactness one may use the differential operator

$$(3.5) \qquad D^i = \left(\frac{\delta}{\delta x} + z\frac{\delta}{\delta y} + f\frac{\delta}{\delta z}\right)^i$$

where $z = y_n'$, $f = f(x_n, y_n, y_n')$ and $\frac{\delta}{\delta x}$, $\frac{\delta}{\delta y}$ and $\frac{\delta}{\delta z}$ are treated as algebraic variables. Thus

$$(3.6) \qquad y_n^{(3)} = \frac{d}{dx} f = f_x + zf_y + ff_z = Df ,$$

$$(3.7) \qquad y_n^{(4)} = \frac{d}{dx} Df = f_{xx} + 2zf_{xy} + 2ff_{xz}$$

$$+ 2zff_{yz} + z^2 f_{yy} + f^2 f_{zz}$$

$$+ f_z(f_x + zf_y + ff_z) + ff_y$$

$$= D^2 f + f_z Df + ff_y$$

and

$$(3.8) \qquad y_n^{(5)} = \frac{d}{dx}(D^2 f + f_z Df + ff_y)$$

$$= f_{xxx} + 3zf_{xxy} + 3ff_{xxz}$$

$$+ z^3 f_{yyy} + 3z^2 f_{xyy} + 3z^2 ff_{yyz}$$

$$+ 6zff_{xyz} + 3f^2 f_{xzz} + 3zf^2 f_{yzz}$$

$$+ f^3 f_{zzz} + 3f(f_{xy}+zf_{yy}+ff_{yz})$$

$$+ 3(f_x+zf_y+ff_z)(f_{xz}+zf_{yz}+ff_{zz})$$

$$+ f_z[f_{xx} + 2zf_{xy} + 2ff_{xz}$$

$$+ 2zff_{yz} + z^2 f_{yy} + f^2 f_{zz}$$

$$+ f_z(f_x+zf_y+ff_z) + ff_y]$$

$$+ f_y(f_x+zf_y+ff_z)$$

$$= D^3 f + 3fDf_y + 3DfDf_z$$

$$+ f_z(D^2 f+f_z Df+ff_y) + f_y Df$$

where

$$f_{t_1 t_2 \ldots t_n} = \frac{\delta^n f}{\delta t_1 \delta t_2 \ldots \delta t_n} . \tag{3.9}$$

The substitution of (3.6-8) into (3.3-4) gives

$$y_{n+1} - y_n - v_n = \frac{h^2}{2!} f + \frac{h^3}{3!} Df \tag{3.10}$$

$$+ \frac{h^4}{4!} (D^2 f+f_z Df+ff_y)$$

$$+ \frac{h^5}{5!} [D^3 f + 3fDf_y + 3DfDf_z$$

$$+ f_z(D^2f+f_zDf+ff_y) + f_yDf]$$

$$+ O(h^6)$$

and

$$v_{n+1} - v_n = h^2\ f + \frac{h^3}{2!}\ Df + \frac{h^4}{3!}\ (D^2f+f_zDf+ff_y) \tag{3.11}$$

$$+ \frac{h^5}{4!} [D^3 f + 3fDf_y + 3DfDf_z$$

$$+ f_z(D^2f+f_zDf+ff_y) + f_yDf]$$

$$+ O(h^6)$$

which are the required expansions for the left sides of (3.1-2) in terms of D.

3.1.2 Expansion of the k_i To obtain the expansions for the right sides of (3.1-2) the Taylor series for a function of three variables about the point (x_n, y_n, y_n') is used. It is given by

$$f(x_n+h_1, y_n+h_2, y_n'+h_3) = \sum_{i=0}^{m^*} \frac{1}{i!} B^i f + O(h^{m^*+1}) \tag{3.12}$$

where

$$B^i = \left(h_1 \frac{\delta}{\delta x} + h_2 \frac{\delta}{\delta y} + h_3 \frac{\delta}{\delta z}\right)^i , \tag{3.13}$$

$z = y_n'$ as defined previously and the exponent has the meaning of repeated application of B. The k_i's for a method of rank four are

$$k_1 = \frac{h^2}{2!} f(x_n, y_n, y_n') = \frac{h^2}{2!} f , \tag{3.14}$$

$$k_2 = \frac{h^2}{2!} f[x_n+a_2h,\ y_n+a_2v_n+b_{21}k_1,\ (v_n+2c_{21}k_1)/h] , \tag{3.15}$$

$$k_3 = \frac{h^2}{2!} f[x_n+a_3h,\ y_n+a_3v_n+b_{31}k_1+b_{32}k_2, \tag{3.16}$$

$$(v_n+2(c_{31}k_1+c_{32}k_2))/h]$$

and

$$k_4 = \frac{h^2}{2!} f[x_n+a_4h,\ y_n+a_4v_n+b_{41}k_1+b_{42}k_2+b_{43}k_3, \tag{3.17}$$

$$(v_n+2(c_{41}k_1+c_{42}k_2+c_{43}k_3))/h] .$$

Since a method of rank four is also of degree four it is necessary, in view of the factor $h^2/2!$, to retain only terms through h^3 when expanding $2!k_i/h^2$. Sub-

stituting (2.12) and (3.14) into (3.15) and expanding yields

$$\text{(3.18)} \quad \frac{2!}{h^2} k_2 = f[x_n+a_2h,\ y_n+a_2hz+\frac{h^2}{2!}b_{21}f,\ z+c_{21}hf]$$

$$= f + [a_2hf_x + a_2hzf_y + \frac{h^2}{2!}b_{21}ff_y$$

$$+ c_{21}hff_z] + \frac{1}{2!} [(a_2h)^2 f_{xx}$$

$$+ (a_2hz)^2 f_{yy} + 2(a_2hz)(\frac{h^2}{2!}b_{21}f)f_{yy}$$

$$+ (c_{21}hf)^2 f_{zz} + 2(a_2h)(a_2hz)f_{xy}$$

$$+ 2(a_2h)(\frac{h^2}{2!}b_{21}f)f_{xy} + 2(a_2h)(c_{21}hf)f_{xz}$$

$$+ 2(a_2hz)(c_{21}hf)f_{yz} + 2(\frac{h^2}{2!}b_{21}f)(c_{21}hf)f_{yz}]$$

$$+ \frac{1}{3!} [(a_2h)^3 f_{xxx} + (a_2hz)^3 f_{yyy} + (c_{21}hf)^3 f_{zzz}$$

$$+ 3(a_2h)^2(a_2hz)f_{xxy} + 3(a_2h)(a_2hz)^2 f_{xyy}$$

$$+ 3(a_2h)^2(c_{21}hf)f_{xxz} + 3(a_2h)(c_{21}hf)^2 f_{xzz}$$

$$+ 3(a_2hz)^2(c_{21}hf)f_{yyz} + 3(a_2hz)(c_{21}hf)^2 f_{yzz}$$

$$+ 6(a_2h)(a_2hz)(c_{21}hf)f_{xyz}] .$$

Gathering terms in $\frac{h^i}{i!}$; $i=0,1,2,3$ gives

$$(3.19) \qquad \frac{2!}{h^2} k_2 = f + h[a_2 f_x + a_2 z f_y + c_{21} f f_z]$$

$$+ \frac{h^2}{2!} [a_2^2 f_{xx} + a_2^2 z^2 f_{yy} + c_{21}^2 f^2 f_{zz}$$

$$+ 2a_2^2 z f_{xy} + 2a_2 c_{21} f f_{xz} + 2a_2 c_{21} z f f_{yz}$$

$$+ b_{21} f f_y] + \frac{h^3}{3!} [a_2^3 f_{xxx} + a_2^3 z^3 f_{yyy}$$

$$+ c_{21}^3 f^3 f_{zzz} + 3a_2^3 z f_{xxy} + 3a_2^3 z^2 f_{xyy}$$

$$+ 3a_2^2 c_{21} f f_{xxz} + 3a_2 c_{21}^2 f f_{xzz}$$

$$+ 3a_2^2 c_{21} z^2 f f_{yyz} + 3a_2 c_{21}^2 z f^2 f_{yzz}$$

$$+ 6a_2^2 c_{21} z f f_{xyz} + 3a_2 b_{21} f f_{xy}$$

$$+ 3a_2 b_{21} z f f_{yy} + 3b_{21} c_{21} f^2 f_{yz}] .$$

Utilizing the operator D as given by (3.5) and the restrictions

$$(3.20) \qquad a_2 = c_{21}; \quad b_{21} = a_2^2$$

one obtains

$$(3.21) \qquad k_2 = \frac{h^2}{2!} f + \frac{h^3}{2!} a_2 Df + \frac{h^4}{2!2!} a_2^2(D^2 f + ff_y)$$

$$+ \frac{h^5}{2!3!} a_2^3(D^3 f + 3fDf_y) + O(h^6) \ .$$

Substitution of (3.14) and (3.21) into (3.16) retaining terms through h^3 gives

$$(3.22) \qquad \frac{2!}{h^2} k_3 = f[x_n + a_3 h,\ y_n + a_3 hz + \frac{h^2}{2!} b_{31} f$$

$$+ b_{32}(\frac{h^2}{2!} f + \frac{h^3}{2!} a_2 Df),\ z + c_{31} hf$$

$$+ c_{32}(hf + a_2 h^2 Df + \frac{h^3}{2!} a_2^2(D^2 f + ff_y))]$$

$$= f[x_n + a_3 h,\ y_n + a_3 hz + (b_{31} + b_{32})\frac{h^2}{2!} f$$

$$+ \frac{h^3}{2!} a_2 b_{32} Df,\ z + (c_{31} + c_{32})hf$$

$$+ a_3 c_{32} h^2 Df + \frac{h^3}{2!} a_2^2 c_{32}(D^2 f + ff_y))] \ .$$

Expanding in the same manner as $2!k_2/h^2$ and imposing the restrictions

$$(3.23) \qquad a_3 = c_{31} + c_{32}; \quad a_3^2 = b_{31} + b_{32}$$

gives in terms of D

$$(3.24) \qquad k_3 = \frac{h^2}{2!} f + \frac{h^3}{2!} a_3 Df + \frac{h^4}{2!2!} [a_3^2(D^2 f + ff_y)$$

$$+ 2a_2 c_{32} f_z Df] + \frac{h^5}{2!3!} [a_3^3(D^3 f + 3fDf_y)$$

$$+ 3a_2^2 c_{32}(f_z D^2 f + ff_y f_z) + 6a_2 a_3 c_{32} DfDf_z$$

$$+ 3a_2 b_{32} f_y Df] .$$

Following a procedure similar to that for k_2 and k_3 with the restrictions

$$(3.25) \qquad a_4 = c_{41} + c_{42} + c_{43}; \quad a_2^2 = b_{41} + b_{42} + b_{43}$$

yields

$$(3.26) \qquad k_4 = \frac{h^2}{2!} f + \frac{h^3}{2!} a_4^2 Df + \frac{h^4}{2!2!} [a_4^2(D^2 f + ff_y)$$

$$+ 2(a_2 c_{42} + a_3 c_{43}) f_z Df] + \frac{h^5}{2!3!} [a_4^3(D^3 f + 3fDf_y)$$

$$+ 3(a_2 b_{42} + a_3 b_{43}) f_y Df + 3(a_2^2 c_{42} + a_3^2 c_{43}) f_z (D^2 f + ff_y)$$

$$+ 6a_4(a_2 c_{42} + a_3 c_{43}) DfDf_z + 6a_2 c_{32} c_{43} f_z^2 Df] .$$

Multiplying (3.14), (3.21), (3.24) and (3.26), respectively, by the corresponding w_{pi}, p=0,1, and gathering terms gives the following, respective, expansions for the right sides of (3.1-2):

(3.27) $$w_{p1}k_1 + w_{p2}k_2 + w_{p3}k_3 + w_{p4}k_4$$

$$= (w_{p1}+w_{p2}+w_{p3}+w_{p4}) \frac{h^2}{2!} f$$

$$+ (a_2w_{p2}+a_3w_{p3}+a_4w_{p4}) \frac{h^3}{2!} Df$$

$$+ (a_2^2w_{p2}+a_3^2w_{p3}+a_4^2w_{p4}) \frac{h^4}{2!2!} (D^2f+ff_y)$$

$$+ 2[a_2c_{32}w_{p3} + (a_2c_{42}+a_3c_{43})w_{p4}] \frac{h^4}{2!2!} f_zDf$$

$$+ (a_2^3w_{p2}+a_3^3w_{p3}+a_4^3w_{p4}) \frac{h^5}{2!3!} (D^3f+3fDf_y)$$

$$+ 3[a_2^2c_{32}w_{p3} + (a_2^2c_{42}+a_3^2c_{43})w_{p4}] \frac{h^5}{2!3!} f_z(D^2f+ff_y)$$

$$+ 3[a_2b_{32}w_{p3} + (a_2b_{42}+a_3b_{43})w_{p4}] \frac{h^5}{2!3!} f_yDf$$

$$+ 6[a_3a_2c_{32}w_{p3} + a_4(a_2c_{42}+a_3c_{43})w_{p4}] \frac{h^5}{2!3!} DfDf_z$$

$$+ 6a_2c_{32}c_{43}w_{p4} \frac{h^5}{2!3!} f_z^2Df$$

where $p = 0$, and

(3.28) same as (3.27) except $p = 1$.

Matching terms in powers of h^2 through h^5 for (3.10) and (3.27), and (3.11) and (3.28) yields the constraints given in matrix form by (3.31). The restrictions used in finding the expansions for the k_i are necessarily included as constraints and are given by

$$a_i = \sum_{j=1}^{i-1} c_{ij}; \quad i=2,3,4, \tag{3.29}$$

and

$$a_i^2 = \sum_{j=1}^{i-1} b_{ij}; \quad i=2,3,4 . \tag{3.30}$$

$$
\begin{array}{l}
\frac{h^2}{2!}f \\[2ex]
\frac{h^3}{3!}Df \\[2ex]
\frac{h^4}{4!}(D^2f+ff_y) \\[2ex]
\frac{h^4}{4!}f_zDf \\[2ex]
\frac{h^5}{5!}(D^3f+3fDf_y) \\[2ex]
\frac{h^5}{5!}f_z(D^2f+ff_y) \\[2ex]
\frac{h^5}{5!}f_yDf \\[2ex]
\frac{h^5}{5!}DfDf_z \\[2ex]
\frac{h^5}{5!}f_z^2Df
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 3a_2 & 3a_3 & 3a_4 \\
0 & 6a_2^2 & 6a_3^2 & 6a_4^2 \\
0 & 0 & 12a_2c_{32} & 12(a_2c_{42}+a_3c_{43}) \\
0 & 10a_2^3 & 10a_3^3 & 10a_4^3 \\
0 & 0 & 30a_2^2c_{32} & 30(a_2^2c_{42}+a_3^2c_{43}) \\
0 & 0 & 30a_2b_{32} & 30(a_2b_{42}+a_3b_{43}) \\
0 & 0 & 60a_3a_2c_{32} & 60a_4(a_2c_{42}+a_3c_{43}) \\
0 & 0 & 0 & 60a_2c_{32}c_{43}
\end{bmatrix}
\begin{bmatrix}
w_{01} & w_{11} \\
w_{02} & w_{12} \\
w_{03} & w_{13} \\
w_{04} & w_{14}
\end{bmatrix}
=
\begin{bmatrix}
1 & 2 \\
1 & 3 \\
1 & 4 \\
1 & 4 \\
1 & 5 \\
1 & 5 \\
1 & 5 \\
1 & 5 \\
1 & 5
\end{bmatrix}
\tag{3.31}
$$

Other derivations for constraints of the RKN method of rank four, notably Nystrom [16] and Duffin [5], excluded the restriction imposed by (3.30). In this instance six other constraints are found as follows:

from $\frac{h^4}{4!} ff_y$

(3.32) $6[b_{21}w_{02} + (b_{31}+b_{32})w_{03} + (b_{41}+b_{42}+b_{43})w_{04}] = 1$,

(3.33) $6[b_{21}w_{12} + (b_{31}+b_{32})w_{13} + (b_{41}+b_{42}+b_{43})w_{14}] = 4$,

from $\frac{h^5}{5!} fDf_y$

(3.34) $30[a_2b_{21}w_{02} + a_3(b_{31}+b_{32})w_{03} + a_4(b_{41}+b_{42}+b_{43})w_{04}] = 1$,

(3.35) $30[a_2b_{21}w_{12} + a_3(b_{31}+b_{32})w_{13} + a_4(b_{41}+b_{42}+b_{43})w_{14}] = 5$,

from $\frac{h^5}{5!} ff_y$

(3.36) $30[b_{21}c_{32}w_{03} + (b_{21}c_{42}+(b_{31}+b_{32})c_{43})]w_{04} = 1$,

(3.37) $30[b_{21}c_{32}w_{13} + (b_{21}c_{42}+(b_{31}+b_{32})c_{43})]w_{14} = 5$,

and would have been included in (3.31). As is easily seen the imposition of (3.30) reduces the number of

constraints involved in a method of rank four from twenty-seven to twenty-four.

3.2 Truncation Error for a Method of Rank Four

The truncation error involved in a RKN method of rank m is given in general by

$$(3.38) \qquad T_m = \frac{h^{m^*+1}}{(m^*+1)!} t_m + O(h^{m^*+2}) .$$

The procedure followed by Lotkin [15] and outlined in Chapter I for a RK method of rank four will be used in obtaining a similar bound on the truncation error for a RKN method of the same rank.

All constraints included in (3.29-31) with the exception of those given by

(3.39)

$$\begin{bmatrix} 0 & 10a_2^3 & 10a_3^3 & 10a_4^3 \\ 0 & 0 & 30a_2^2c_{32} & 30(a_2^2c_{42}+a_3^2c_{43} \\ 0 & 0 & 30a_2b_{32} & 30(a_2b_{42}+a_3b_{43}) \\ 0 & 0 & 60a_3a_2c_{32} & 60a_4(a_2c_{42}+a_3c_{43}) \\ 0 & 0 & 0 & 60a_2c_{32}c_{43} \end{bmatrix} \begin{bmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

must be satisfied for a RKN method of rank four. It is these constraints which determine the value of t_m in (3.38) for $m = m^* = 4$.

Throughout the derivation of the constraints in Section 3.1 terms in h^5 were retained, not only because necessary constraints were derived from them, but also to obtain a bound on $|t_4|$. To obtain an estimate of the truncation error involved it is necessary to find the difference between terms in $h^5/5!$ in (3.10) and (3.27). This is given by

$$(3.40) \qquad t_4 = u_1(D^3f+3fDf_y) + u_2f_z(D^2f+ff_y) + u_3f_yDf + u_4DfDf_z + u_5f_z^2Df ,$$

where

$$(3.41) \qquad u_1 = 1 - 10(a_2^3w_{02}+a_3^3w_{03}+a_4^3w_{04}) ,$$

$$(3.42) \qquad u_2 = 1 - 30[a_2^2c_{32}w_{03} + (a_2^2c_{42}+a_3^2c_{43})w_{04}] ,$$

$$(3.43) \qquad u_3 = 1 - 30[a_2b_{32}w_{03} + (a_2b_{42}+a_3b_{43})w_{04}] ,$$

$$(3.44) \qquad u_4 = 1 - 60[a_3a_2c_{32}w_{03} + a_4(a_2c_{42}+a_3c_{43})w_{04}]$$

and

$$(3.45) \qquad u_5 = 1 - 60(a_2 c_{32} c_{43} w_{04}) \;.$$

The evaluation of t_4 by (3.40) is dependent upon $f(x,y,y')$ while the purpose of Lotkin's procedure is to obtain a bound on the truncation error independent of this function. This is accomplished by defining the following in a region R containing all points in (2.24):

$$(3.46) \qquad \begin{gathered} |f(x,y,y')| < M; \quad |y'| = |z| < N \\ \left| \frac{\delta^{i+j+k} f}{\delta x^i \delta y^j \delta z^k} \right| < \frac{L^{i+j+k}}{N^j M^{k-1}} \end{gathered}$$

where $i + j + k \leq 4$ and L, M and N are positive constants independent of x and y.

Using (3.46) enables limits to be placed on the terms in (3.40) dependent on $f(x,y,y')$. The basic elements in these terms are given below with their respective limits.

$$(3.47) \qquad \begin{array}{lll} f_x < LM & f_{xx} < L^2M & f_{xxx} < L^3M \\ f_y < LMN^{-1} & f_{yy} < L^2MN^{-2} & f_{yyy} < L^3MN^{-3} \\ f_z < L & f_{zz} < L^2M^{-1} & f_{zzz} < L^3M^{-2} \\ f_{xy} < L^2MN^{-1} & f_{xz} < L^2 & f_{yz} < L^2N^{-1} \\ f_{xxy} < L^3MN^{-1} & f_{xyy} < L^3MN^{-2} & f_{xxz} < L^3 \\ f_{xzz} < L^3M^{-1} & f_{yyz} < L^3N^{-2} & f_{yzz} < L^3M^{-1}N^{-1} \end{array}$$

The seven coefficients of the u_i on the right side of (3.40) may be rewritten without D as

$$(3.48)\qquad D^3f = f_{xxx} + z^3f_{yyy} + f^3f_{zzz} + 3zf_{xxy} + 3ff_{xxz} + 3z^2f_{xyy} + 3f^2f_{xzz} + 3z^2ff_{yyz} + 3zf^2f_{yzz} + 6zff_{xyz} ,$$

$$(3.49)\qquad fDf_y = ff_{xy} + zff_{yy} + f^2f_{yz} ,$$

$$(3.50)\qquad f_zD^2f = f_zf_{xx} + z^2f_zf_{yy} + f^2f_zf_{zz} + 2zf_zf_{xy} + 2ff_zf_{xz} + 2zff_zf_{yz} ,$$

$$(3.51)\qquad ff_yf_z = ff_yf_z ,$$

$$(3.52)\qquad f_yDf = f_xf_y + zf_y^2 + ff_yf_z ,$$

$$(3.53)\qquad DfDf_z = f_xf_{xz} + zf_xf_{yz} + ff_xf_{zz} + zf_yf_{xz} + z^2f_yf_{yz} + zff_yf_{zz} + ff_zf_{xz} + zff_zf_{yz} + f^2f_zf_{zz}$$

and

$$(3.54) \qquad f_z^2 Df = f_z^2 f_x + z f_z^2 f_y + f f_z^3 \, .$$

Collecting terms on the right sides of (3.48-54) and using their respective limits, easily obtained from (3.47), the following absolute bound on (3.40) is obtained:

$$(3.55) \quad |t_4| < (27|u_1| + 4|u_2| + |u_2+u_4| + 2|2u_2+u_4| + 6|u_4| + 3|u_5|)L^3M + (9|u_1| + |u_2+u_3| + 2|u_3|)L^2M^2N^{-1} \, .$$

For a RKN method of rank four the absolute value of (3.38) can then be written as

$$(3.56) \qquad |T_4| < h^5(e_1L^3M + e_2L^2M^2N^{-1}) + O(h^6)$$

where

$$(3.57) \qquad e_1 = (27|u_1|+4|u_2|+|u_2+u_4|+2|2u_2+u_4| +6|u_4|+3|u_5|)/5! \, ,$$

and

$$e_2 = (9|u_1|+|u_2+u_3|+2|u_3|)/5! \; . \tag{3.58}$$

A second order ODE often encountered is that given by

$$y'' = f(x,y); \quad y'(x_0) = y_0'; \quad y(x_0) = y_0 \tag{3.59}$$

in which the first derivative y' does not appear explicitly. In this case, since any partial derivative with respect to z is equal to zero, the expression for the truncation error involved in solving (3.59) by a RKN method is significantly different than that given by (3.56-58). If all terms are ignored which contain a partial derivative with respect to z it is found that

$$|t_4| < 8|u_1|L^3M + (6|u_1|+2|u_3|)L^2M^2N^{-1} \tag{3.60}$$

and

$$|T_4|_{f(x,y)} < \frac{h^5}{5!}[8|u_1|L^3M + (6|u_1|+2|u_3|)]L^2M^2N^{-1} + O(h^6) \; . \tag{3.61}$$

3.3 The RKN Method of Rank Three

Using the expansions for the left and right sides of (3.1-2) derived in Section 3.1 for $m = 4$ the

corresponding expansions for the RKN method of rank three can be obtained simply by retaining only terms through h^4. From (3.10-11)

$$(3.62) \qquad y_{n+1} - y_n - v_n = \frac{h^2}{2!}f + \frac{h^3}{3!}Df + \frac{h^4}{4!}(D^2f+f_zDf+ff_y) + O(h^5)$$

and

$$(3.63) \qquad v_{n+1} - v_n = h^2f + \frac{h^3}{2!}Df + \frac{h^4}{3!}(D^2f+f_zDf+ff_y) + O(h^5) ,$$

while (3.27-28) gives

$$(3.64\text{-}65) \qquad \begin{aligned} &w_{p1}k_1 + w_{p2}k_2 + w_{p3}k_3 \\ &= (w_{p1}+w_{p2}+w_{p3})\ \frac{h^2}{2!}f \\ &+ (a_2w_{p2}+a_3w_{p3})\ \frac{h^3}{2!}Df \\ &+ (a_2^2w_{p2}+a_2^2w_{p3})\ \frac{h^4}{2!2!}(D^2f+ff_y) \\ &+ 2a_2c_{32}w_{p3}\ \frac{h^4}{2!2!}f_zDf \end{aligned}$$

where again $p = 0,1$.

3.3.1 <u>The Constraints for a RKN Method of Rank Three</u> Matching terms in powers of h^2 through h^4 for (3.62) and (3.64), and (3.63) and (3.65) yields the necessary constraints given in matrix form by

$$(3.66) \qquad \begin{array}{l} \frac{h^2}{2!}f \\[2ex] \frac{h^3}{3!}Df \\[2ex] \frac{h^4}{4!}(D^2f+ff_y) \\[2ex] \frac{h^4}{4!}f_zDf \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3a_2 & 3a_3 \\ 0 & 6a_2^2 & 6a_3^2 \\ 0 & 0 & 12a_2c_{32} \end{bmatrix} \begin{bmatrix} w_{01} & w_{11} \\ w_{02} & w_{12} \\ w_{03} & w_{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}.$$

The necessary restrictions

$$(3.67) \qquad a_i = \sum_{j=1}^{i-1} c_{ij}; \qquad i=2,3$$

and

$$(3.68) \qquad a_i^2 = \sum_{j=1}^{i-1} b_{ij}; \qquad i=2,3$$

must be included with (3.66) to comprise the full set of constraints for a method of rank three. If (3.68) is not imposed two other constraints are found from the term $(h^4/4!)ff_y$ and are given by

(3.69) $$6[b_{21}w_{02} + (b_{31}+b_{32})w_{03}] = 1$$

and

(3.70) $$6[b_{21}w_{12} + (b_{31}+b_{32})w_{13}] = 4 \ .$$

The imposition of (3.68) does not reduce the number of constraints involved in a method of rank three.

3.3.2 <u>The Truncation Error</u> The value of t_m in (3.38) for $m = 3$ is obtained by finding the difference between terms in $h^4/4!$ in (3.62) and (3.64). This is given by

(3.71) $$t_3 = u_1(D^2f+ff_y) + u_2f_zDf$$

where

(3.72) $$u_1 = 1 - 6(a_2^2w_{02}+a_3^2w_{03})$$

and

(3.73) $$u_2 = 1 - 12a_2c_{32}w_{03} \ .$$

Following a similar procedure to that used in Section 3.2, and employing (3.47), the following bound

can be placed on the absolute value of t_3.

(3.74) $$|t_3| < (9|u_1|+3|u_2|)L^2M + |u_1|LM^2N^{-1} .$$

It is then possible to write (3.38) for RKN methods of rank three as

(3.75) $$|T_3| < h^4(e_1L^2M + e_2LM^2N^{-1}) + O(h^5)$$

where

(3.76) $$e_1 = (9|u_1|+3|u_2|)/4!$$

and

(3.77) $$e_2 = |u_1|/4! \quad .$$

For an ODE of type (3.59), the bound on the truncation error for $m = 3$ is given by

(3.78) $$|T_3|_{f(x,y)} < \frac{h^4}{4!}(4|u_1|L^2M + |u_1|LM^2N^{-1}) + O(h^5)$$

which was found by again considering all partial derivatives with respect to z as being equal to zero.

CHAPTER IV

NEW SETS OF COEFFICIENTS FOR RUNGE-KUTTA-NYSTROM METHODS OF RANK FOUR

4.1 Two RKN Methods

The set of coefficients for RKN methods most widely used for computational purposes is that proposed by Nystrom [16] and given by

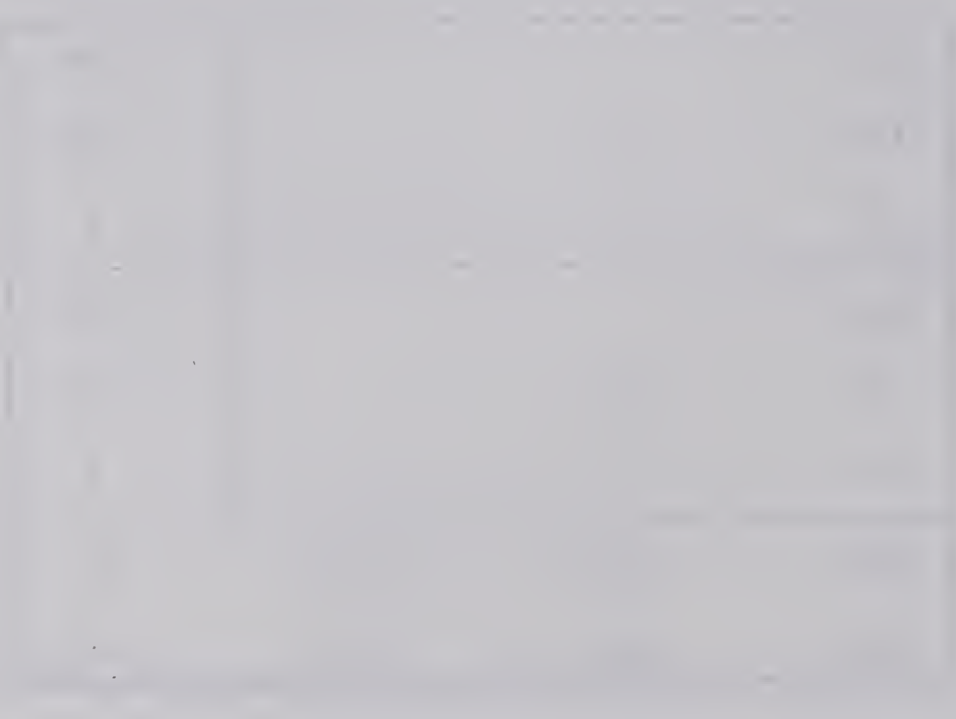
(4.1)

1/4			1/4
1/4	0		1/4
0	0	1	1
1/2			1/2
0	1/2		1/2
0	0	1	1
1/3	1/3	1/3	0
1/3	2/3	2/3	1/3

The bound on the truncation error involved using (4.1) is

$$(4.2) \qquad |T_4| < h^5(0.0979\ L^3M + 0.0354\ L^2M^2N^{-1}) + O(h^6)$$

calculated from (3.56). The bound obtained for a problem of type (3.59) is found from (3.61) and given by



$$(4.3)\quad |T_4|_{f(x,y)} < h^5(0.0111\ L^3M + 0.025\ L^2M^2N^{-1}) + O(h^6)\ .$$

Another set of coefficients was suggested by Duffin [5] and given by

(4.4)

1/9			1/9
1/9	1/3		4/9
1/4	1/2	1/4	1
1/3			1/3
-1/3	1		2/3
1	-1	1	1
1/4	1/2	1/4	0
1/4	3/4	3/4	1/4

where the coefficients e_1 and e_2 for the truncation error bounds are

(4.5)

y''	e_1	e_2
$f(x,y,y')$	0.0653	0.0111
$f(x,y)$	0.00494	0.00648

Application of (4.4) to several second order ODE's produced better results than those obtained using (4.1) or the repeated application of any RK method of rank four.

Superior stability was also indicated. These results were to be expected since the error bounds for (4.4) were smaller than those for (4.1) [5].

4.2 The Two Sets of Constraints

The constraints originally derived by Nystrom [16] and Duffin [5] for a RKN method of rank four are given by (3.29), (3.31) and (3.32-37) which form a system of twenty-seven non-linear equations in twenty-three unknowns. As seven of these equations, given by (3.34), (3.36) and (3.39), comprise the bound on the truncation error, the actual system necessary to be solved is reduced to twenty equations in twenty-three unknowns. Attempts were made to solve this three parameter system but were unsuccessful.

A close examination of the constraints derived by Nystrom and Duffin revealed that if restriction (3.30) were imposed, the complete system would be reduced to one of twenty-four equations in twenty-three unknowns given by (3.29-31). The system to be solved would not include the five constraints of (3.39) and would be a four parameter system of nineteen equations in twenty-three unknowns. Although another degree of freedom was obtained, attempts to solve this system were also unsuccessful.

4.3 Two Procedures for Obtaining New RKN Methods

Any set of constraints for a RKN method of rank four obtained from a RK method of the same rank using (2.25) satisfies all necessary constraints derived by Nystrom and Duffin. They do not, however, satisfy the restrictions imposed by (3.30) in the new derivation of constraints.

4.3.1 Procedure I

To obtain coefficients for RKN methods of rank four one can obviously use (2.25). Applying such a RKN method to solve a second order ODE is the same as using the corresponding RK method to solve an equivalent system of first order equations. To improve upon these coefficients, in terms of truncation error bound and the results obtained in the solution of second order equations, it is noted that, in the new derivation, only four of the nineteen necessary constraints contain terms in b_{ij}. These are given by

(4.6) $$a_2^2 = b_{21} ,$$

(4.7) $$a_3^2 = b_{31} + b_{32} ,$$

(4.8) $$a_4^2 = b_{41} + b_{42} + b_{43}$$

and

(4.9) $$30a_2b_{32}w_{13} + 30(a_2b_{42}+a_3b_{43})w_{14} = 5 \ .$$

If one first determines all coefficients from (2.25) except the b_{ij}; $j < i$, it is possible to consider (4.7-9) as a two parameter system of three equations in the five unknowns b_{31}, b_{32}, b_{41}, b_{42} and b_{43}. The coefficient b_{21} can be determined from (4.6).

If b_{42} and b_{43} are chosen as the two parameters one obtains from (4.9)

(4.10) $$b_{32} = \frac{1 - 6(a_2b_{42}+a_3b_{43})w_{14}}{6a_2w_{13}} \ .$$

Hence (4.6-8) yield

(4.11) $$b_{21} = a_2^2 \ ,$$

(4.12) $$b_{31} = a_3^2 - b_{32}$$

and

(4.13) $$b_{41} = a_4^2 - b_{42} - b_{43} \ .$$

Using (4.10-13), values for the b_{ij} can be obtained

which satisfy all necessary constraints in (3.29-31).

Both (4.1) and (4.4) can be obtained from (2.25) and (4.10-13). Using the classical RK method given in Appendix I, with $b_{42} = 0$ and $b_{43} = 1$, Procedure I gives (4.1). From Boulton's RK method also in Appendix I, with $b_{42} = \frac{1}{2}$ and $b_{43} = \frac{1}{4}$, the above procedure yields (4.4).

4.3.2 Procedure II A second procedure to determine new RKN methods is to include a fifth constraint in determining the b_{ij} while again obtaining values for the other coefficients as in Procedure I. The constraint

$$(4.14) \qquad 30a_2b_{32}w_{03} + 30(a_2b_{42}+a_3b_{43})w_{04} = 1$$

found in (3.39) contains terms in b_{ij} and the inclusion of it enables (4.7-9) and (4.14) to be treated as a one parameter system of four equations in five unknowns. Again b_{21} can be determined separately from (4.6).

Any four unknowns may be expressed in terms of the remaining one. If b_{43} is chosen, (4.9) and (4.14) yield

$$(4.15) \qquad b_{32} = \frac{5w_{04} - w_{14}}{30a_2(w_{04}w_{13}-w_{03}w_{14})}$$

and

$$(4.16) \qquad b_{42} = \frac{(5w_{03}-w_{13}) - 30a_3b_{43}(w_{03}w_{14}-w_{04}w_{13})}{30a_2(w_{03}w_{14}-w_{04}w_{13})} .$$

The remaining b_{ij} may be determined from (4.11-13).

The use of either procedure reduces the bounds on the truncation error given by (3.56) and (3.61) compared to that using coefficients obtained only from (2.25). The reduction is in the term e_2 for both bounds and is larger using Procedure II since the use of (4.14) as a constraint reduces $|u_3|$, given by (3.43), to zero.

4.4 New Runge-Kutta-Nystrom Methods

Consider the RK method of rank four proposed by Kuntzmann [3] and given by

(4.17)

2/5			2/5
-3/20	15/20		3/5
19/44	-15/44	40/44	1
11/72	25/72	25/72	11/72

Determining the corresponding coefficients for a RKN method from (2.25) yields

(4.18)

0			4/25
3/5	0		9/25
-6/11	15/11	0	1
2/5			2/5
-3/20	15/20		3/5
19/44	-15/44	40/44	1
11/36	15/36	10/36	0
11/36	25/36	25/36	11/36

Any set of coefficients determined in this manner will be referred to as the X RKN method where X is the name of the RK set of coefficients; i.e. (4.18) is the Kuntzmann RKN method. If the b_{ij} for a RKN method are determined by either Procedure I or II, the method will be referred to as a X RKN J where X is as defined previously and J is equal to I or II.

The b_{ij} of the Kuntzmann RKN I where $b_{42} = 3/5$ and $b_{43} = 1/5$ are given by

(4.19)

4/25			4/25
17/250	73/250		90/250
1/5	3/5	1/5	1

The corresponding b_{ij} of the Kuntzmann RKN II where $b_{43} = 1/5$ are

$$\begin{array}{|ccc|c|} \hline 4/25 & & & 4/25 \\ 3/50 & 15/50 & & 18/50 \\ 18/55 & 21/55 & 11/55 & 1 \\ \hline \end{array} \qquad (4.20)$$

4.4.1 <u>Truncation Error Bounds</u> The particular choices for b_{42} and b_{43} were made for both Kuntzmann RKN I and II because they gave the most substantial decrease in the truncation error bounds, not only for Kuntzmann's RK method, but also for all others used. This was determined by trying different values for the parameters within the interval [0,1].

As all X RKN methods are determined using (2.25), the truncation error bounds for these methods can be considered equivalent to the corresponding bounds using the X RK method. Because of this one can use this bound as a measure in comparing the error bounds of X RKN J methods as opposed to the use of X RK methods in solving a second order ODE.

The coefficients e_1 and e_2 of the truncation error bounds for Kuntzmann's RKN methods, as well as Nystrom's, Duffin's and those listed in Appendix I, are all given in Table 4.1, illustrating the fact that, for any given X, e_2 is smallest for X RKN II methods.

TABLE 4.1

Truncation Error Bounds for

Runge-Kutta-Nystrom Methods

	$f(x,y,y')$		$f(x,y)$	
	e_1	e_2	e_1	e_2
Ralston RKN	0.120	0.0514	0.0121	0.0287
Ralston RKN I	0.120	0.0169	0.0121	0.0112
Ralston RKN II	0.120	0.0136	0.0121	0.00907
Nystrom (4.1)	0.0979	0.0354	0.0111	0.025
Classical RKN	0.0979	0.0458	0.0111	0.025
Classical RKN I	0.0979	0.0230	0.0111	0.0167
Classical RKN II	0.0979	0.0146	0.0111	0.00833
Gill RKN	0.0979	0.0458	0.0111	0.025
Gill RKN I	0.0979	0.0230	0.0111	0.0167
Gill RKN II	0.0979	0.0146	0.0111	0.00833
Duffin (4.4)	0.0653	0.0111	0.00494	0.00648
Boulton RKN	0.0653	0.0417	0.00494	0.0222
Boulton RKN I	0.0653	0.0111	0.00494	0.00648
Boulton RKN II	0.0653	0.00694	0.00494	0.00370
Kuntzmann RKN	0.055	0.0333	0.00889	0.0167
Kuntzmann RKN I	0.055	0.018	0.00889	0.012
Kuntzmann RKN II	0.055	0.01	0.00889	0.00667

A close examination of Table 4.1 reveals there is no method that always has the lowest values of both e_1 and e_2. Using the sum of e_1 and e_2 as a criterion, although this is not necessarily significant, it is obvious that among the methods listed in Table 4.1 the Kuntzmann RKN II method has the smallest bound when $y'' = f(x,y,y')$, and the Boulton RKN II method for $f(x,y)$.

CHAPTER V

NUMERICAL RESULTS

5.1 Test Problems

To assess the relative merits and demerits of the X RKN J methods in solving second order ODE's numerical solutions of several problems with known analytical solutions were computed by use of both RK and RKN methods. These test problems are

$$(5.1) \qquad y'' = y; \quad y(0) = y'(0) = 1 ,$$

$$(5.2) \qquad y'' = -0.5 + 1.5y'; \quad y(0) = y'(0) = 1 ,$$

$$(5.3) \qquad y'' = 11y - 10y'; \quad y(0) = y'(0) = 1 ,$$

$$(5.4) \qquad y'' = y; \quad y(0) = 1, \ y'(0) = -1 ,$$

$$(5.5) \qquad y'' = -0.5y - 1.5y'; \quad y(0) = 1, \ y'(0) = -1 ,$$

$$(5.6) \qquad y'' = 11y + 10y'; \quad y(0) = 1, \ y'(0) = -1 ,$$

$$(5.7) \qquad y'' = y; \quad y(0) = 2, \ y'(0) = 0 ,$$

$$(5.8) \qquad y'' = y - 2 \sin x; \quad y(0) = 1, \ y'(0) = 0 ,$$

(5.9) $$y'' = -y; \quad y(0) = 0,\ y'(0) = 1$$

and

(5.10) $$y'' = -y; \quad y(0) = 1,\ y'(0) = 0\ .$$

Their corresponding analytical solutions are given by $y = e^x$ for (5.1-3); $y = e^{-x}$ for (5.4-6); $y = e^x + e^{-x}$ for (5.7); $y = \sin x + e^{-x}$ for (5.8); $y = \sin x$ for (5.9); and $y = \cos x$ for (5.10). The above problems are often used in the literature [3], [4] and most have exponential solutions since this type of problem is best suited to measure accuracy and stability.

All problems are solved, both as a simultaneous system of first order equations using Ralston's RK method of rank four, and as a second order ODE by Ralston RKN II and Kuntzmann or Boulton RKN II methods of corresponding rank (Appendix I). These particular methods are used because Ralston RK (R1) method has the minimum truncation error bound for a first order equation while Kuntzmann RKN II (K2) has the corresponding bound for second order equations where $y'' = f(x,y,y')$, and Boulton RKN II (B2) for $y'' = f(x,y)$. The numerical results are listed in Appendix III.

Comparisons in accuracy and stability are made for all test problems by listing the ratios of the actual errors involved for each method. In this manner the RKN methods are compared with the RK methods used as well as with each other.

Several other initial value problems with analytical solutions involving logarithmic and trigonometric functions were solved. The results, although not listed in Appendix III, favoured the use of the X RKN J methods.

5.2 Accuracy

The results of problems (5.1), (5.4) and (5.7-10) indicate the superiority of RKN methods over RK methods in terms of initial accuracy. For these six problems the X RKN J methods give up to three more significant digits of accuracy than the Ralston RK method; i.e. problem (5.1), (5.4), (5.7) and (5.10).

The Boulton RKN II method, used for the above mentioned problems because all are of the form $y'' = f(x,y)$, gives better results than the Ralston RKN II method. For (5.7) and (5.10) the difference is as much as two significant figures initially while for (5.1), (5.4) and (5.8-9) the difference is not as significant.

The second order ODE's given by (5.2-3) and (5.5-6) are of the form $y'' = f(x,y,y')$, hence the Kuntzmann RKN II method was used. These problems were specifically constructed to illustrate what can occur in solving differential equations numerically. In all four problems the Ralston RK method gives better results than the X RKN II methods used. A close examination of the computational procedure followed for these problems indicates the results can be attributed to a significant build-up in round-off error. This is caused by the subtraction of two numbers of approximately the same magnitude and occurs for the RKN methods because of the restrictions on the coefficients imposed by (3.30). The propagation of round-off error is magnified in these four problems because of the form of the function $f(x,y,y')$, and is especially noticeable in the solutions of (5.3) and (5.6).

The Ralston RKN II method gives better results for (5.2-3) while Kuntzmann's is superior for (5.5-6). However, the differences in these results are insignificant.

5.3 Stability

Problems (5.1) and (5.4) were integrated over a larger interval than other problems to indicate the stability of the different methods. The results for (5.1) indicate not

only superior accuracy for the X RKN II methods, but this accuracy is maintained throughout the interval of integration. Conversely, although the solution for (5.4) is initially more accurate, the accumulation of round-off error is significant and results in instability for this problem using X RKN II methods. The Ralston RK method maintains stability throughout the interval as in (5.1).

The significance of the round-off error in (5.4) can be attributed to restriction (3.30) and the fact that y, y' and y'' are of equal absolute value. This causes the subtraction of numbers of approximately the same magnitude.

The instability as well as inaccuracy for the solutions of (5.2-3) and (5.5-6) can also be attributed to restriction (3.30) and the values of y, y' and y''. The form of the function $f(x,y,y')$ magnifies significantly the accumulation of error in these problems. The obvious differences in accuracy and stability for (5.1-3) and (5.4-6) illustrate the sensitivity of the solution to the form of the function $f(x,y,y')$. These four problems maintain stability using the Ralston RK method.

The ODE's given by (5.7-10) have stable as well as accurate solutions throughout the interval of integration for both the Ralston RK and the X RKN II methods used.

There was no particular distinction between the stability of the X RKN II methods used for the test problems.

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 Conclusions

The theoretical and numerical results of this study lead to the following conclusions:

1. The X RKN J methods of rank four obtained from Procedures I and II are superior to the use of the corresponding RK method of the same rank with regards to the truncation error bound found by the approach of Lotkin [15].
2. The X RKN II methods are generally more accurate than the corresponding RK methods of the same rank.
3. The X RKN J methods appear to be more susceptible to the propagation of round-off error than the RK methods. In the sample problems tested this was attributed to the added restrictions imposed on the coefficients by (3.30) and the fact that the absolute magnitudes of y, y' and y'' were equal.
4. The use of either a RK or RKN method of corresponding rank involves the same number of operations and function evaluations for each

integration step if the methods are programmed efficiently with respect to time and round-off error.

5. No single X RKN II method appears best suited for all problems although the best results were obtained from the methods attributed to Ralston, Kuntzmann and Boulton. The method attributed to Boulton is that best suited for functions of the form $f(x,y)$.
6. RKN methods have all the advantages and disadvantages listed in Section 1.3 for RK methods and thus are best used only in obtaining starting values for the solution by predictor-corrector methods.

6.2 Suggestions for Further Research

1. The general relationships between coefficients of RK and RKN methods of the same rank given by (2.25) provide the basis for continuing the research previously described for RKN methods of degree greater than four. Perhaps a general procedure could be developed to obtain modified coefficients for a RKN method of any degree.
2. The general direct Runge-Kutta type methods for third and higher ordered ODE's could be derived in the same manner as the RKN method in Chapter II.

Relationships could then be established between the coefficients for these methods and for the RK methods for first order equations.

3. A third or higher ordered equation can be treated as any system of lower order ODE's, hence, direct RK type methods could not only be derived from a system of first order equations as in Chapter II, but also from some system of ODE's containing equations of higher order than one. For example, a third order equation could be treated as a system consisting of a first and second order ODE while a fourth order equation could, perhaps, be treated as a pair of second order, or first and third order ODE's. In this manner general relationships could be derived between the coefficients of different RK type methods.

4. The two parameter non-linear system for RK methods of rank four has been solved, as mentioned in Chapter I, in terms of α_2 and α_3. Hence, Procedure's I and II have effectively solved the non-linear system necessary for coefficients of RKN methods of rank four in terms of four and three parameters respectively. Perhaps one could express the coefficients of RKN methods of rank four, as well as methods for higher ordered

equations of the same rank, in terms of a few parameters. This would obviously enable a more comprehensive study of such methods.

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ZAMM	Zeitschrift für Angewandte Mathematik und Mechanik
ZAMP	Zeitschrift für Angewandte Mathematik und Physik

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APPENDIX I

SOME RUNGE-KUTTA-NYSTROM METHODS OF RANK FOUR

This appendix contains the coefficients for the best known and most widely used RK methods of rank four for first order ODE's, and the corresponding RKN coefficients obtained using (2.25), Procedure I and Procedure II.

The methods listed are the Classical and those due to Boulton, Gill [10] and Ralston [18]. The RK methods are presented first and labelled RK while the coefficients obtained using only (2.25) are listed under RKN. The b_{ij} found from Procedures I and II are labelled PI and PII, respectively, and listed along with the corresponding values of the a_i^2.

The RKN methods presented by Nystrom [16] and Duffin [5] are given in Chapter IV with the RK and RKN methods attributed to Kuntzmann [3].

CLASSICAL

RK

0.5			0.5
0	0.5		0.5
0	0	1	1
0.16666667	0.33333333	0.33333333	0.16666667

RKN

0			0.25
0.5	0		0.25
0	1	0	1
0.5			0.5
0	0.5		0.5
0	0	1	1
0.33333333	0.33333333	0.33333333	0
0.33333333	0.66666667	0.66666667	0.33333333

PI

0.25			0.25
0.15	0.1		0.25
0.2	0.6	0.2	1

PII

0.25			0.25
0.05	0.2		0.25
0.4	0.4	0.2	1

BOULTON

RK

0.33333333			0.33333333
¯0.33333333	1		0.66666667
1	¯1	1	1
0.125	0.375	0.375	0.125

RKN

0			0.11111111
0.66666667	0		0.44444444
¯1.3333333	2	0	1
0.33333333			0.33333333
¯0.33333333	1		0.66666667
1	¯1	1	1
0.25	0.5	0.25	0
0.25	0.75	0.75	0.25

PI

0.11111111			0.11111111
0.11111111	0.33333333		0.44444444
0.2	0.6	0.2	1

PII

0.11111111			0.11111111
0.044444444	0.4		0.44444444
0.4	0.4	0.2	1

GILL

RK

0.5			0.5
0.20710678	0.29289322	0	0.5
0	¯0.70710678	1.7071068	1
0.16666667	0.097631073	0.56903559	0.16666667

RKN

0			0.25
0.29289322	0		0.25
0	1	0	1
0.5			0.5
0.20710678	0.29289322		0.5
0	¯0.70710678	1.7071068	1
0.33333333	0.097631073	0.56903559	0
0.33333333	0.19526215	1.1380712	0.33333333

PI

0.25			0.25
0.19142136	0.058578644		0.25
0.2	0.6	0.2	1

PII

0.25			0.25
0.13284271	0.11715729		0.25
0.4	0.4	0.2	1

RALSTON

RK

0.4			0.4
0.2969776	0.15875966		0.45573726
0.21810038	¯3.0509647	3.8328643	1
0.17476028	¯0.55148053	1.2055355	0.17118478

RKN

0			0.16
0.12700773	0		0.20769645
¯0.16422207	1.2170085	0	1
0.4			0.4
0.2969776	0.15875966		0.45573726
0.21810038	¯3.0509647	3.8328643	1
0.34952057	¯0.66177664	1.3122561	0
0.34952056	¯1.1029611	2.4110709	0.34236956

PI

0.16			0.16
0.15243899	0.055257457		0.20769645
0.2	0.6	0.2	1

PII

0.16			0.16
0.14419258	0.063503866		0.20769645
0.25807373	0.54192627	0.2	1

APPENDIX II

APL FUNCTION LISTINGS

APL is a machine executable version, available in a time-sharing environment, of what is more commonly known as the Iverson Language. An interpreter has been written for the IBM 360, Models 50 and higher. This particular language was chosen for the numerical experiments because: 1) all computations are performed in double precision; 2) the time-sharing environment provided a means of comprehensively testing all methods and problems, comparing results as they were obtained. For a complete introduction to this language see Introduction to APL 360 Concepts by W.S. Adams, Department of Computing Science, University of Alberta.

The following pages contain listings of the more important functions used in testing the *X RK* and *X RKN J* methods.

*USE*1 *X* and *USE*2 *X* convert the coefficients for RK and RKN methods, respectively, into the proper vector and matrix form necessary for use in solving a problem. The parameter *X* is a vector of length thirteen for *USE*1 and of length twenty-three for *USE*2, the number of coefficients not equal to zero in RK and RKN methods, respectively, of rank four.

*X RK*1 *Y* will solve any simultaneous system of first order equations using a RK method of any rank. *YIPRIME* is used to evaluate the functions of the system being solved and *YEXACT* to calculate the analytical solution.

*X RK*2 *Y* solves second order ODE's directly using RKN methods. *Y2PRIME* is used to evaluate the function involved and *YEXACT* to calculate the analytical solution.

RATIO X calculates the ratio of the actual errors involved in the different methods. These errors have been previously calculated and saved in *RK*1 and *RK*2.

The parameter *X* used in *RK*1, *RK*2 and *RATIO* is a four element vector where *X*[1]←x_0, *X*[2]←h, *X*[3]← "an upper limit on the interval of integration", and *X*[4]←m, the rank of the method.

The parameter *Y* is a vector of length n in *RK*1 where n is the number of equations in the system being solved, and of length two in *RK*2, i.e. *Y*←y_0, y'_0.

```
      ∇USE1[⎕]∇

    ∇ USE1 X
[1]   B1← 4 4 ρA1←W1←4ρ0
[2]   A1[1+ι3]←X[ι3]
[3]   B1[2;1]←X[4]
[4]   B1[3;ι2]←X[4+ι2]
[5]   B1[4;ι3]←X[6+ι3]
[6]   W1[ι4]←X[9+ι4]
    ∇


      ∇USE2[⎕]∇

    ∇ USE2 X
[1]   B2←C2← 4 4 ρW2← 2 4 ρA2←4ρ0
[2]   A2[1+ι3]←X[ι3]
[3]   B2[2;1]←X[4]
[4]   B2[3;ι2]←X[4+ι2]
[5]   B2[4;ι3]←X[6+ι3]
[6]   C2[2;1]←X[10]
[7]   C2[3;ι2]←X[10+ι2]
[8]   C2[4;ι3]←X[12+ι3]
[9]   W2[ι2;]← 2 4 ρX[15+ι8]
    ∇


      ∇RK1[⎕]∇

    ∇ M←X RK1 Y;I;J;K;L
[1]   X[3]←(X[3]-X[1])÷X[2]
[2]   SAVE←M←ιI←0
[3]   K←((ρY),X[4])ρJ←0
[4]   K[;J]←X[2]×(X[1]+A1[J]×X[2])Y1PRIME(Y++/(((ρY),X[
      4])ρB1[(J←J+1);])×K)
[5]   →(J<X[4])/4
[6]   I←I+1
[7]   M←M,X[1],Y,|((ρY)YEXACT X[1]←X[1]+X[2])-Y←Y++/(((ρY),
      X[4])ρW1)×K
[8]   →(X[3]>I)/3
[9]   M←(X[3],1+2×ρY)ρM
[10]  M[; 3 5]←X[2]×M[; 3 5]
[11]  SAVE←SAVE,M[;4],M[;5]
[12]  (6ρ' '),'X',(12ρ' '),'Y',(12ρ' '),'V',(10ρ' '),'|E(Y)
      |',(7ρ' '),'|E(V)|'
    ∇
```

```
      ∇RK2[⎕]∇

    ∇ M←X RK2 Y;I;J;K;L;XV;YV
[1]   X[3]←(X[3]-X[1])÷X[2]
[2]   SAVE←M←⍳I←0
[3]   K←X[4]⍴J←0
[4]   XV←X[1]+A2[J←J+1]×X[2]
[5]   YV←(Y[1]+(A2[J]×X[2]×Y[2])++/B2[J;]×K),(Y[2]++/(
      2÷X[2])×C2[J;]×K)
[6]   K[J]←((X[2]*2)÷2)×XV Y2PRIME YV
[7]   →(J<X[4])/4
[8]   Y[1]←Y[1]+(X[2]×Y[2])++/W2[1;]×K
[9]   Y[2]←Y[2]++/(W2[2;]×K)÷X[2]
[10]  I←I+1
[11]  YV←(⍴Y)YEXACT X[1]←X[1]+X[2]
[12]  M←M,X[1],Y[1],(X[2]×Y[2]),(|YV[1]-Y[1]),|(X[
      2]×YV[2])-X[2]×Y[2]
[13]  →(X[3]>I)/3
[14]  M←(X[3],5)⍴M
[15]  SAVE←SAVE,M[;4],M[;5]
[16]  (6⍴' '),'X',(12⍴' '),'Y',(12⍴' '),'V',(10⍴' '),'|E(Y)
      |',(7⍴' '),'|E(V)|'
    ∇

      ∇RATIO[⎕]∇

    ∇ M←RATIO X;I
[1]   X[4]←(X[3]-X[1])÷X[2]
[2]   S1←S÷S1
[3]   S2←S÷S2
[4]   S←S2÷S1
[5]   M←(X[4],6)⍴0
[6]   M[;1]←S1[⍳X[4]]
[7]   M[;2]←S1[X[4]+⍳X[4]]
[8]   M[;3]←S2[⍳X[4]]
[9]   M[;4]←S2[X[4]+⍳X[4]]
[10]  M[;5]←S[⍳X[4]]
[11]  M[;6]←S[X[4]+⍳X[4]]
[12]  (26⍴' '),'RATIO OF ERRORS'
[13]  ' '
[14]  (6⍴' '),'E(R1)/E(B2)',(11⍴' '),'E(R1)/E(R2)',(
      11⍴' '),'E(B2)/E(R2)'
[15]  (5⍴' '),'Y',(10⍴' '),'V',(10⍴' '),'Y',(10⍴' '),'V',(
      10⍴' '),'Y',(10⍴' '),'V'
    ∇
```

APPENDIX III

LISTING OF NUMERICAL RESULTS

This appendix contains the numerical results found using the Ralston RK (R1) and RKN II (R2) methods to solve a second order ODE. The Boulton (B2) or Kuntzmann (K2) RKN II method is also used in solving the problem. The results are compared by means of a table of the ratio of the actual errors. Listed are the ratios of the errors involved in the Ralston RK method over the Ralston RKN II and the Boulton or Kuntzmann RKN II methods; and the Boulton or Kuntzmann RKN II method over the Ralston RKN II method.

The APL functions *Y1PRIME*, *Y2PRIME* and *YEXACT* are listed with the results, as well as the parameter *X* and *Y* of the functions *RK*1, *RK*2 and *RATIO* (Appendix II). The problems are solved in the order of listing in Chapter V; i.e. (5.1-10). Most are solved over the interval [0,2] although (5.1) and (5.4) are integrated from [0,6]. The interval h is in all cases equal to 0.2.

(5.1)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],Y[1]
    ∇

      X
0   0.2   6   4
      Y
1   1

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.22140E0	2.44280E¯1	2.75818E¯6	5.51635E¯7
4.00000E¯1	1.49182E0	2.98364E¯1	6.73768E¯6	1.34754E¯6
6.00000E¯1	1.82211E0	3.64421E¯1	1.23441E¯5	2.46882E¯6
8.00000E¯1	2.22552E0	4.45104E¯1	2.01028E¯5	4.02057E¯6
1.00000E0	2.71825E0	5.43650E¯1	3.06920E¯5	6.13841E¯6
1.20000E0	3.32007E0	6.64014E¯1	4.49848E¯5	8.99695E¯6
1.40000E0	4.05514E0	8.11027E¯1	6.41018E¯5	1.28204E¯5
1.60000E0	4.95294E0	9.90589E¯1	8.94789E¯5	1.78958E¯5
1.80000E0	6.04952E0	1.20990E0	1.22951E¯4	2.45902E¯5
2.00000E0	7.38889E0	1.47778E0	1.66858E¯4	3.33717E¯5
2.20000E0	9.02479E0	1.80496E0	2.24181E¯4	4.48362E¯5
2.40000E0	1.10229E1	2.20458E0	2.98707E¯4	5.97414E¯5
2.60000E0	1.34633E1	2.69267E0	3.95245E¯4	7.90490E¯5
2.80000E0	1.64441E1	3.28883E0	5.19887E¯4	1.03977E¯4
3.00000E0	2.00849E1	4.01697E0	6.80348E¯4	1.36070E¯4
3.20000E0	2.45316E1	4.90633E0	8.86376E¯4	1.77275E¯4
3.40000E0	2.99629E1	5.99259E0	1.15029E¯3	2.30057E¯4
3.60000E0	3.65967E1	7.31935E0	1.48760E¯3	2.97521E¯4
3.80000E0	4.46993E1	8.93985E0	1.91790E¯3	3.83581E¯4
4.00000E0	5.45957E1	1.09191E1	2.46582E¯3	4.93164E¯4
4.20000E0	6.66832E1	1.33366E1	3.16235E¯3	6.32469E¯4
4.40000E0	8.14468E1	1.62894E1	4.04642E¯3	8.09285E¯4
4.60000E0	9.94791E1	1.98958E1	5.16696E¯3	1.03339E¯3
4.80000E0	1.21504E2	2.43008E1	6.58532E¯3	1.31706E¯3
5.00000E0	1.48405E2	2.96810E1	8.37845E¯3	1.67569E¯3
5.20000E0	1.81262E2	3.62523E1	1.06428E¯2	2.12856E¯3
5.40000E0	2.21393E2	4.42786E1	1.34991E¯2	2.69982E¯3
5.60000E0	2.70409E2	5.40819E1	1.70985E¯2	3.41969E¯3
5.80000E0	3.30278E2	6.60556E1	2.16299E¯2	4.32599E¯3
6.00000E0	4.03401E2	8.06803E1	2.73298E¯2	5.46597E¯3

```
      S←SAVE
```

<u>(5.1)</u>

```
      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←Y[1]
    ∇

      USE2 B2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.22140E0	2.44281E¯1	2.60461E¯9	7.27648E¯9
4.00000E¯1	1.49182E0	2.98365E¯1	1.31632E¯8	1.64149E¯8
6.00000E¯1	1.82212E0	3.64424E¯1	3.38376E¯8	2.81296E¯8
8.00000E¯1	2.22554E0	4.45108E¯1	6.75799E¯8	4.33152E¯8
1.00000E0	2.71828E0	5.43656E¯1	1.18337E¯7	6.30997E¯8
1.20000E0	3.32012E0	6.64023E¯1	1.91313E¯7	8.89106E¯8
1.40000E0	4.05520E0	8.11040E¯1	2.93304E¯7	1.22557E¯7
1.60000E0	4.95303E0	9.90606E¯1	4.33128E¯7	1.66335E¯7
1.80000E0	6.04965E0	1.20993E0	6.22166E¯7	2.23154E¯7
2.00000E0	7.38906E0	1.47781E0	8.75052E¯7	2.96705E¯7
2.20000E0	9.02501E0	1.80500E0	1.21054E¯6	3.91661E¯7
2.40000E0	1.10232E1	2.20463E0	1.65262E¯6	5.13936E¯7
2.60000E0	1.34637E1	2.69275E0	2.23186E¯6	6.71005E¯7
2.80000E0	1.64446E1	3.28893E0	2.98720E¯6	8.72309E¯7
3.00000E0	2.00855E1	4.01711E0	3.96812E¯6	1.12976E¯6
3.20000E0	2.45325E1	4.90650E0	5.23736E¯6	1.45837E¯6
3.40000E0	2.99641E1	5.99282E0	6.87447E¯6	1.87704E¯6
3.60000E0	3.65982E1	7.31964E0	8.98003E¯6	2.40955E¯6
3.80000E0	4.47012E1	8.94023E0	1.16812E¯5	3.08581E¯6
4.00000E0	5.45981E1	1.09196E1	1.51385E¯5	3.94337E¯6
4.20000E0	6.66863E1	1.33373E1	1.95542E¯5	5.02936E¯6
4.40000E0	8.14508E1	1.62902E1	2.51832E¯5	6.40292E¯6
4.60000E0	9.94843E1	1.98969E1	3.23464E¯5	8.13814E¯6
4.80000E0	1.21510E2	2.43021E1	4.14471E¯5	1.03278E¯5
5.00000E0	1.48413E2	2.96826E1	5.29921E¯5	1.30882E¯5
5.20000E0	1.81272E2	3.62544E1	6.76177E¯5	1.65646E¯5
5.40000E0	2.21406E2	4.42813E1	8.61220E¯5	2.09388E¯5
5.60000E0	2.70426E2	5.40853E1	1.09506E¯4	2.64379E¯5
5.80000E0	3.30299E2	6.60599E1	1.39022E¯4	3.33457E¯5
6.00000E0	4.03429E2	8.06857E1	1.76240E¯4	4.20163E¯5

```
      S1←SAVE
```

(5.1)

USE2 R2
X RK2 Y

X	Y	V	$\lvert E(Y)\rvert$	$\lvert E(V)\rvert$
2.00000E¯1	1.22140E0	2.44281E¯1	1.51582E¯8	1.49063E¯9
4.00000E¯1	1.49182E0	2.98365E¯1	3.24760E¯8	2.73082E¯9
6.00000E¯1	1.82212E0	3.64424E¯1	5.29920E¯8	3.70166E¯9
8.00000E¯1	2.22554E0	4.45108E¯1	7.79490E¯8	4.35820E¯9
1.00000E0	2.71828E0	5.43656E¯1	1.08861E¯7	4.62432E¯9
1.20000E0	3.32012E0	6.64023E¯1	1.47595E¯7	4.38553E¯9
1.40000E0	4.05520E0	8.11040E¯1	1.96468E¯7	3.47937E¯9
1.60000E0	4.95303E0	9.90606E¯1	2.58378E¯7	1.68275E¯9
1.80000E0	6.04965E0	1.20993E0	3.36947E¯7	1.30450E¯9
2.00000E0	7.38906E0	1.47781E0	4.36724E¯7	5.88084E¯9
2.20000E0	9.02501E0	1.80500E0	5.63412E¯7	1.25702E¯8
2.40000E0	1.10232E1	2.20464E0	7.24175E¯7	2.20565E¯8
2.60000E0	1.34637E1	2.69275E0	9.28002E¯7	3.52282E¯8
2.80000E0	1.64446E1	3.28893E0	1.18617E¯6	5.32337E¯8
3.00000E0	2.00855E1	4.01711E0	1.51284E¯6	7.75530E¯8
3.20000E0	2.45325E1	4.90651E0	1.92573E¯6	1.10087E¯7
3.40000E0	2.99641E1	5.99282E0	2.44706E¯6	1.53271E¯7
3.60000E0	3.65982E1	7.31965E0	3.10466E¯6	2.10217E¯7
3.80000E0	4.47012E1	8.94024E0	3.93334E¯6	2.84897E¯7
4.00000E0	5.45982E1	1.09196E1	4.97666E¯6	3.82366E¯7
4.20000E0	6.66863E1	1.33373E1	6.28906E¯6	5.09050E¯7
4.40000E0	8.14509E1	1.62902E1	7.93855E¯6	6.73103E¯7
4.60000E0	9.94843E1	1.98969E1	1.00101E¯5	8.84860E¯7
4.80000E0	1.21510E2	2.43021E1	1.26097E¯5	1.15740E¯6
5.00000E0	1.48413E2	2.96826E1	1.58698E¯5	1.50726E¯6
5.20000E0	1.81272E2	3.62545E1	1.99552E¯5	1.95531E¯6
5.40000E0	2.21406E2	4.42813E1	2.50718E¯5	2.52788E¯6
5.60000E0	2.70426E2	5.40853E1	3.14758E¯5	3.25814E¯6
5.80000E0	3.30300E2	6.60599E1	3.94865E¯5	4.18786E¯6
6.00000E0	4.03429E2	8.06858E1	4.95014E¯5	5.36955E¯6

S2←SAVE

SIGDIG 4

6

<u>(5.1)</u>

```
      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(*X),*X
    ∇

      RATIO X
```

RATIO OF ERRORS

E(R1)/E(B2)		E(R1)/E(R2)		E(B2)/E(R2)	
Y	V	Y	V	Y	V
1.059E3	7.581E1	1.820E2	3.701E2	1.718E¯1	4.881E0
5.119E2	8.209E1	2.075E2	4.935E2	4.053E¯1	6.011E0
3.648E2	8.777E1	2.329E2	6.670E2	6.385E¯1	7.599E0
2.975E2	9.282E1	2.579E2	9.225E2	8.670E¯1	9.939E0
2.594E2	9.728E1	2.819E2	1.327E3	1.087E0	1.365E1
2.351E2	1.012E2	3.048E2	2.052E3	1.296E0	2.027E1
2.186E2	1.046E2	3.263E2	3.685E3	1.493E0	3.522E1
2.066E2	1.076E2	3.463E2	1.063E4	1.676E0	9.885E1
1.976E2	1.102E2	3.649E2	1.885E4	1.846E0	1.711E2
1.907E2	1.125E2	3.821E2	5.675E3	2.004E0	5.045E1
1.852E2	1.145E2	3.979E2	3.567E3	2.149E0	3.116E1
1.807E2	1.162E2	4.125E2	2.709E3	2.282E0	2.330E1
1.771E2	1.178E2	4.259E2	2.244E3	2.405E0	1.905E1
1.740E2	1.192E2	4.383E2	1.953E3	2.518E0	1.639E1
1.715E2	1.204E2	4.497E2	1.755E3	2.623E0	1.457E1
1.692E2	1.216E2	4.603E2	1.610E3	2.720E0	1.325E1
1.673E2	1.226E2	4.701E2	1.501E3	2.809E0	1.225E1
1.657E2	1.235E2	4.792E2	1.415E3	2.892E0	1.146E1
1.642E2	1.243E2	4.876E2	1.346E3	2.970E0	1.083E1
1.629E2	1.251E2	4.955E2	1.290E3	3.042E0	1.031E1
1.617E2	1.258E2	5.028E2	1.242E3	3.109E0	9.880E0
1.607E2	1.264E2	5.097E2	1.202E3	3.172E0	9.513E0
1.597E2	1.270E2	5.162E2	1.168E3	3.231E0	9.197E0
1.589E2	1.275E2	5.222E2	1.138E3	3.287E0	8.923E0
1.581E2	1.280E2	5.279E2	1.112E3	3.339E0	8.683E0
1.574E2	1.285E2	5.333E2	1.089E3	3.388E0	8.472E0
1.567E2	1.289E2	5.384E2	1.068E3	3.435E0	8.283E0
1.561E2	1.293E2	5.432E2	1.050E3	3.479E0	8.114E0
1.556E2	1.297E2	5.478E2	1.033E3	3.521E0	7.962E0
1.551E2	1.301E2	5.521E2	1.018E3	3.560E0	7.825E0

```
      SIGDIG 6
4
```

(5.2)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],(-0.5×Y[1])+1.5×Y[2]
    ∇

      X
0  0.2  2  4
      Y
1  1

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.22140E0	2.44280E¯1	2.75818E¯6	5.51635E¯7
4.00000E¯1	1.49182E0	2.98364E¯1	6.73768E¯6	1.34754E¯6
6.00000E¯1	1.82211E0	3.64421E¯1	1.23441E¯5	2.46882E¯6
8.00000E¯1	2.22552E0	4.45104E¯1	2.01028E¯5	4.02057E¯6
1.00000E0	2.71825E0	5.43650E¯1	3.06920E¯5	6.13841E¯6
1.20000E0	3.32007E0	6.64014E¯1	4.49848E¯5	8.99695E¯6
1.40000E0	4.05514E0	8.11027E¯1	6.41018E¯5	1.28204E¯5
1.60000E0	4.95294E0	9.90589E¯1	8.94789E¯5	1.78958E¯5
1.80000E0	6.04952E0	1.20990E0	1.22951E¯4	2.45902E¯5
2.00000E0	7.38889E0	1.47778E0	1.66858E¯4	3.33717E¯5

```
      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Ỳ
[1]   V←(-0.5×Y[1])+1.5×Y[2]
    ∇

      USE2 K2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.22140E0	2.44279E¯1	6.06483E¯6	1.60836E¯6
4.00000E¯1	1.49181E0	2.98361E¯1	1.52747E¯5	3.97485E¯6
6.00000E¯1	1.82209E0	3.64416E¯1	2.87733E¯5	7.36116E¯6
8.00000E¯1	2.22549E0	4.45096E¯1	4.80616E¯5	1.21082E¯5
1.00000E0	2.71821E0	5.43638E¯1	7.51007E¯5	1.86585E¯5
1.20000E0	3.32000E0	6.63996E¯1	1.12442E¯4	2.75842E¯5
1.40000E0	4.05504E0	8.11000E¯1	1.63395E¯4	3.96234E¯5
1.60000E0	4.95280E0	9.90551E¯1	2.32236E¯4	5.57251E¯5
1.80000E0	6.04932E0	1.20985E0	3.24475E¯4	7.71069E¯5
2.00000E0	7.38861E0	1.47771E0	4.47196E¯4	1.05327E¯4

```
      S1←SAVE
```

<u>(5.2)</u>

```
      USE2 R2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.22140E0	2.44279E¯1	6.06484E¯6	1.45728E¯6
4.00000E¯1	1.49181E0	2.98361E¯1	1.50992E¯5	3.58824E¯6
6.00000E¯1	1.82209E0	3.64417E¯1	2.81504E¯5	6.62272E¯6
8.00000E¯1	2.22549E0	4.45097E¯1	4.65874E¯5	1.08597E¯5
1.00000E0	2.71821E0	5.43640E¯1	7.21917E¯5	1.66864E¯5
1.20000E0	3.32001E0	6.63999E¯1	1.07273E¯4	2.46033E¯5
1.40000E0	4.05505E0	8.11005E¯1	1.54819E¯4	3.52547E¯5
1.60000E0	4.95281E0	9.90557E¯1	2.18677E¯4	4.94682E¯5
1.80000E0	6.04934E0	1.20986E0	3.03793E¯4	6.83045E¯5
2.00000E0	7.38864E0	1.47772E0	4.16513E¯4	9.31198E¯5

```
      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(*X),*X
    ∇

      RATIO X
```

RATIO OF ERRORS

E(R1)/E(K2)		E(R1)/E(R2)		E(K2)/E(R2)	
Y	V	Y	V	Y	V
0.4548	0.343	0.4548	0.3785	1	1.104
0.4411	0.339	0.4462	0.3755	1.012	1.108
0.429	0.3354	0.4385	0.3728	1.022	1.112
0.4183	0.3321	0.4315	0.3702	1.032	1.115
0.4087	0.329	0.4251	0.3679	1.04	1.118
0.4001	0.3262	0.4193	0.3657	1.048	1.121
0.3923	0.3236	0.414	0.3637	1.055	1.124
0.3853	0.3211	0.4092	0.3618	1.062	1.126
0.3789	0.3189	0.4047	0.36	1.068	1.129
0.3731	0.3168	0.4006	0.3584	1.074	1.131

```
      SIGDIG 6
4
```

(5.3)

```
        ∇Y1PRIME[⎕]∇
      ∇ V←X Y1PRIME Y
[1]     V←Y[2],(11×Y[1])-10×Y[2]
      ∇

        X
0   0.2   2   4
        Y
1   1

        USE1 R1
        X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.22140E0	2.44280E¯1	2.75818E¯6	5.51635E¯7
4.00000E¯1	1.49182E0	2.98364E¯1	6.73768E¯6	1.34754E¯6
6.00000E¯1	1.82211E0	3.64421E¯1	1.23441E¯5	2.46882E¯6
8.00000E¯1	2.22552E0	4.45104E¯1	2.01028E¯5	4.02057E¯6
1.00000E0	2.71825E0	5.43650E¯1	3.06920E¯5	6.13841E¯6
1.20000E0	3.32007E0	6.64014E¯1	4.49848E¯5	8.99695E¯6
1.40000E0	4.05514E0	8.11027E¯1	6.41018E¯5	1.28204E¯5
1.60000E0	4.95294E0	9.90589E¯1	8.94789E¯5	1.78958E¯5
1.80000E0	6.04952E0	1.20990E0	1.22951E¯4	2.45902E¯5
2.00000E0	7.38889E0	1.47778E0	1.66858E¯4	3.33717E¯5

```
        S←SAVE

        ∇Y2PRIME[⎕]∇
      ∇ V←X Y2PRIME Y
[1]     V←(11×Y[1])-10×Y[2]
      ∇

        USE2 K2
        X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
0.2	1.22115	0.244809	0.000253851	0.000528847
0.4	1.49142	0.299185	0.000409276	0.000819951
0.6	1.82157	0.36548	0.000550561	0.00105619
0.8	2.22483	0.446412	0.000710759	0.00130404
1	2.71737	0.545249	0.000907014	0.00159221
1.2	3.31896	0.665962	0.00115269	0.00193815
1.4	4.05374	0.813397	0.00146182	0.00235726
1.6	4.95118	0.993473	0.00185102	0.00286628
1.8	6.04731	1.21341	0.00234081	0.00348493
2	7.3861	1.48205	0.0029567	0.00423694

```
        S1←SAVE
```

(5.3)

```
      USE2 R2
      X RK2 Y

      X             Y             V             |E(Y)|        |E(V)|

 0.2          1.22115       0.244838      0.000253851   0.000557316
 0.4          1.49143       0.299239      0.000398924   0.000874106
 0.6          1.8216        0.365558      0.00051898    0.00113462
 0.8          2.22489       0.446517      0.000645967   0.00140888
 1            2.71749       0.545385      0.000794482   0.00172855

 1.2          3.31914       0.666137      0.000973878   0.00211363
 1.4          4.05401       0.813622      0.00119265    0.00258203
 1.6          4.95157       0.99376       0.00146017    0.00315337
 1.8          6.04786       1.21378       0.00178755    0.00385085
 2            7.38687       1.48251       0.00218827    0.00470249

      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(*X),*X
    ∇

      RATIO X

                          RATIO OF ERRORS

      E(R1)/E(K2)             E(R1)/E(R2)             E(K2)/E(R2)
    Y           V           Y           V           Y           V

 0.01087    0.001043    0.01087     0.0009898   1           0.9489
 0.01646    0.001643    0.01689     0.001542    1.026       0.938
 0.02242    0.002337    0.02379     0.002176    1.061       0.9309
 0.02828    0.003083    0.03112     0.002854    1.1         0.9256
 0.03384    0.003855    0.03863     0.003551    1.142       0.9211

 0.03903    0.004642    0.04619     0.004257    1.184       0.917
 0.04385    0.005439    0.05375     0.004965    1.226       0.9129
 0.04834    0.006244    0.06128     0.005675    1.268       0.909
 0.05253    0.007056    0.06878     0.006386    1.31        0.905
 0.05643    0.007876    0.07625     0.007097    1.351       0.901

      SIGDIG 6
4
```

(5.4)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],Y[1]
    ∇

      X
0  0.2  6  4
      Y
1  ¯1

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.18733E¯1	¯1.63747E¯1	2.58026E¯6	5.16052E¯7
4.00000E¯1	6.70324E¯1	¯1.34065E¯1	4.22509E¯6	8.45017E¯7
6.00000E¯1	5.48817E¯1	¯1.09763E¯1	5.18882E¯6	1.03776E¯6
8.00000E¯1	4.49335E¯1	¯8.98669E¯2	5.66434E¯6	1.13287E¯6
1.00000E0	3.67885E¯1	¯7.35770E¯2	5.79697E¯6	1.15939E¯6
1.20000E0	3.01200E¯1	¯6.02400E¯2	5.69540E¯6	1.13908E¯6
1.40000E0	2.46602E¯1	¯4.93205E¯2	5.44017E¯6	1.08803E¯6
1.60000E0	2.01902E¯1	¯4.03803E¯2	5.09034E¯6	1.01807E¯6
1.80000E0	1.65304E¯1	¯3.30607E¯2	4.68857E¯6	9.37715E¯7
2.00000E0	1.35340E¯1	¯2.70679E¯2	4.26521E¯6	8.53041E¯7
2.20000E0	1.10807E¯1	¯2.21614E¯2	3.84127E¯6	7.68253E¯7
2.40000E0	9.07214E¯2	¯1.81443E¯2	3.43087E¯6	6.86175E¯7
2.60000E0	7.42766E¯2	¯1.48553E¯2	3.04305E¯6	6.08609E¯7
2.80000E0	6.08127E¯2	¯1.21625E¯2	2.68309E¯6	5.36618E¯7
3.00000E0	4.97894E¯2	¯9.95788E¯3	2.35364E¯6	4.70728E¯7
3.20000E0	4.07643E¯2	¯8.15285E¯3	2.05547E¯6	4.11094E¯7
3.40000E0	3.33751E¯2	¯6.67501E¯3	1.78806E¯6	3.57611E¯7
3.60000E0	2.73253E¯2	¯5.46505E¯3	1.55005E¯6	3.10011E¯7
3.80000E0	2.23721E¯2	¯4.47442E¯3	1.33958E¯6	2.67917E¯7
4.00000E0	1.83168E¯2	¯3.66336E¯3	1.15448E¯6	2.30897E¯7
4.20000E0	1.49966E¯2	¯2.99931E¯3	9.92474E¯7	1.98495E¯7
4.40000E0	1.22782E¯2	¯2.45564E¯3	8.51264E¯7	1.70253E¯7
4.60000E0	1.00526E¯2	¯2.01051E¯3	7.28637E¯7	1.45727E¯7
4.80000E0	8.23037E¯3	¯1.64607E¯3	6.22496E¯7	1.24499E¯7
5.00000E0	6.73848E¯3	¯1.34770E¯3	5.30893E¯7	1.06179E¯7
5.20000E0	5.51702E¯3	¯1.10340E¯3	4.52045E¯7	9.04091E¯8
5.40000E0	4.51697E¯3	¯9.03393E¯4	3.84339E¯7	7.68677E¯8
5.60000E0	3.69819E¯3	¯7.39638E¯4	3.26325E¯7	6.52650E¯8
5.80000E0	3.02783E¯3	¯6.05566E¯4	2.76715E¯7	5.53429E¯8
6.00000E0	2.47899E¯3	¯4.95797E¯4	2.34367E¯7	4.68735E¯8

```
      S←SAVE
```

(5.4)

```
      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←Y[1]
    ∇

      USE2 B2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.18731E¯1	¯1.63746E¯1	2.47757E¯9	6.97115E¯9
4.00000E¯1	6.70320E¯1	¯1.34064E¯1	1.15735E¯8	1.29183E¯8
6.00000E¯1	5.48812E¯1	¯1.09762E¯1	2.64711E¯8	1.83165E¯8
8.00000E¯1	4.49329E¯1	¯8.98658E¯2	4.68008E¯8	2.35758E¯8
1.00000E0	3.67880E¯1	¯7.35759E¯2	7.25865E¯8	2.90657E¯8
1.20000E0	3.01194E¯1	¯6.02388E¯2	1.04214E¯7	3.51364E¯8
1.40000E0	2.46597E¯1	¯4.93194E¯2	1.42423E¯7	4.21376E¯8
1.60000E0	2.01897E¯1	¯4.03793E¯2	1.88311E¯7	5.04372E¯8
1.80000E0	1.65299E¯1	¯3.30597E¯2	2.43364E¯7	6.04395E¯8
2.00000E0	1.35336E¯1	¯2.70670E¯2	3.09500E¯7	7.26042E¯8
2.20000E0	1.10804E¯1	¯2.21605E¯2	3.89135E¯7	8.74673E¯8
2.40000E0	9.07184E¯2	¯1.81435E¯2	4.85270E¯7	1.05664E¯7
2.60000E0	7.42742E¯2	¯1.48546E¯2	6.01603E¯7	1.27958E¯7
2.80000E0	6.08108E¯2	¯1.21619E¯2	7.42672E¯7	1.55268E¯7
3.00000E0	4.97880E¯2	¯9.95722E¯3	9.14030E¯7	1.88713E¯7
3.20000E0	4.07633E¯2	¯8.15221E¯3	1.12247E¯6	2.29652E¯7
3.40000E0	3.33746E¯2	¯6.67437E¯3	1.37628E¯6	2.79743E¯7
3.60000E0	2.73254E¯2	¯5.46440E¯3	1.68559E¯6	3.41008E¯7
3.80000E0	2.23728E¯2	¯4.47374E¯3	2.06277E¯6	4.15916E¯7
4.00000E0	1.83182E¯2	¯3.66262E¯3	2.52291E¯6	5.07480E¯7
4.20000E0	1.49987E¯2	¯2.99850E¯3	3.08446E¯6	6.19382E¯7
4.40000E0	1.22811E¯2	¯2.45471E¯3	3.76991E¯6	7.56118E¯7
4.60000E0	1.00564E¯2	¯2.00944E¯3	4.60676E¯6	9.23180E¯7
4.80000E0	8.23538E¯3	¯1.64482E¯3	5.62857E¯6	1.12728E¯6
5.00000E0	6.74482E¯3	¯1.34621E¯3	6.87634E¯6	1.37660E¯6
5.20000E0	5.52496E¯3	¯1.10163E¯3	8.40014E¯6	1.68116E¯6
5.40000E0	4.52684E¯3	¯9.01263E¯4	1.02611E¯5	2.05319E¯6
5.60000E0	3.71040E¯3	¯7.37065E¯4	1.25339E¯5	2.50761E¯6
5.80000E0	3.04286E¯3	¯6.02448E¯4	1.53098E¯5	3.06266E¯6
6.00000E0	2.49745E¯3	¯4.92010E¯4	1.87002E¯5	3.74062E¯6

```
      S1←SAVE
```

<u>(5.4)</u>

```
USE2 R2
X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.18731E¯1	¯1.63746E¯1	2.02637E¯8	2.85255E¯8
4.00000E¯1	6.70320E¯1	¯1.34064E¯1	6.59769E¯8	5.32687E¯8
6.00000E¯1	5.48812E¯1	¯1.09762E¯1	1.34509E¯7	7.61155E¯8
8.00000E¯1	4.49329E¯1	¯8.98657E¯2	2.24953E¯7	9.87144E¯8
1.00000E0	3.67880E¯1	¯7.35758E¯2	3.37946E¯7	1.22571E¯7
1.20000E0	3.01195E¯1	¯6.02387E¯2	4.75571E¯7	1.49132E¯7
1.40000E0	2.46598E¯1	¯4.93192E¯2	6.41346E¯7	1.79867E¯7
1.60000E0	2.01897E¯1	¯4.03791E¯2	8.40281E¯7	2.16336E¯7
1.80000E0	1.65300E¯1	¯3.30595E¯2	1.07902E¯6	2.60272E¯7
2.00000E0	1.35337E¯1	¯2.70667E¯2	1.36603E¯6	3.13659E¯7
2.20000E0	1.10805E¯1	¯2.21603E¯2	1.71194E¯6	3.78819E¯7
2.40000E0	9.07201E¯2	¯1.81431E¯2	2.12988E¯6	4.58517E¯7
2.60000E0	7.42762E¯2	¯1.48542E¯2	2.63604E¯6	5.56070E¯7
2.80000E0	6.08133E¯2	¯1.21613E¯2	3.25023E¯6	6.75493E¯7
3.00000E0	4.97911E¯2	¯9.95659E¯3	3.99669E¯6	8.21660E¯7
3.20000E0	4.07671E¯2	¯8.15144E¯3	4.90505E¯6	1.00050E¯6
3.40000E0	3.33793E¯2	¯6.67343E¯3	6.01149E¯6	1.21926E¯6
3.60000E0	2.73311E¯2	¯5.46326E¯3	7.36019E¯6	1.48674E¯6
3.80000E0	2.23798E¯2	¯4.47234E¯3	9.00512E¯6	1.81373E¯6
4.00000E0	1.83267E¯2	¯3.66091E¯3	1.10121E¯5	2.21337E¯6
4.20000E0	1.50090E¯2	¯2.99641E¯3	1.34616E¯5	2.70174E¯6
4.40000E0	1.22938E¯2	¯2.45217E¯3	1.64518E¯5	3.29844E¯6
4.60000E0	1.00719E¯2	¯2.00634E¯3	2.01027E¯5	4.02745E¯6
4.80000E0	8.25431E¯3	¯1.64103E¯3	2.45607E¯5	4.91804E¯6
5.00000E0	6.76795E¯3	¯1.34158E¯3	3.00046E¯5	6.00595E¯6
5.20000E0	5.55322E¯3	¯1.09598E¯3	3.66529E¯5	7.33486E¯6
5.40000E0	4.56135E¯3	¯8.94358E¯4	4.47723E¯5	8.95811E¯6
5.60000E0	3.75255E¯3	¯7.28632E¯4	5.46888E¯5	1.09409E¯5
5.80000E0	3.09436E¯3	¯5.92148E¯4	6.68003E¯5	1.33627E¯5
6.00000E0	2.56034E¯3	¯4.79430E¯4	8.15927E¯5	1.63208E¯5

```
S2←SAVE

SIGDIG 4
```

6

(5.4)

```
      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(*-X),-*-X
    ∇

      RATIO X
```

RATIO OF ERRORS

E(R1)/E(B2)		E(R1)/E(R2)		E(B2)/E(R2)	
Y	V	Y	V	Y	V
1.041E3	7.403E1	1.273E2	1.809E1	1.223E¯1	2.444E¯1
3.651E2	6.541E1	6.404E1	1.586E1	1.754E¯1	2.425E¯1
1.960E2	5.666E1	3.858E1	1.363E1	1.968E¯1	2.406E¯1
1.210E2	4.805E1	2.518E1	1.148E1	2.080E¯1	2.388E¯1
7.986E1	3.989E1	1.715E1	9.459E0	2.148E¯1	2.371E¯1
5.465E1	3.242E1	1.198E1	7.638E0	2.191E¯1	2.356E¯1
3.820E1	2.582E1	8.482E0	6.049E0	2.221E¯1	2.343E¯1
2.703E1	2.018E1	6.058E0	4.706E0	2.241E¯1	2.331E¯1
1.927E1	1.551E1	4.345E0	3.603E0	2.255E¯1	2.322E¯1
1.378E1	1.175E1	3.122E0	2.720E0	2.266E¯1	2.315E¯1
9.871E0	8.783E0	2.244E0	2.028E0	2.273E¯1	2.309E¯1
7.070E0	6.494E0	1.611E0	1.497E0	2.278E¯1	2.304E¯1
5.058E0	4.756E0	1.154E0	1.094E0	2.282E¯1	2.301E¯1
3.613E0	3.456E0	8.255E¯1	7.944E¯1	2.285E¯1	2.299E¯1
2.575E0	2.494E0	5.889E¯1	5.729E¯1	2.287E¯1	2.297E¯1
1.831E0	1.790E0	4.191E¯1	4.109E¯1	2.288E¯1	2.295E¯1
1.299E0	1.278E0	2.974E¯1	2.933E¯1	2.289E¯1	2.294E¯1
9.196E¯1	9.091E¯1	2.106E¯1	2.085E¯1	2.290E¯1	2.294E¯1
6.494E¯1	6.442E¯1	1.488E¯1	1.477E¯1	2.291E¯1	2.293E¯1
4.576E¯1	4.550E¯1	1.048E¯1	1.043E¯1	2.291E¯1	2.293E¯1
3.218E¯1	3.205E¯1	7.373E¯2	7.347E¯2	2.291E¯1	2.293E¯1
2.258E¯1	2.252E¯1	5.174E¯2	5.162E¯2	2.291E¯1	2.292E¯1
1.582E¯1	1.579E¯1	3.625E¯2	3.618E¯2	2.292E¯1	2.292E¯1
1.106E¯1	1.104E¯1	2.535E¯2	2.531E¯2	2.292E¯1	2.292E¯1
7.721E¯2	7.713E¯2	1.769E¯2	1.768E¯2	2.292E¯1	2.292E¯1
5.381E¯2	5.378E¯2	1.233E¯2	1.233E¯2	2.292E¯1	2.292E¯1
3.746E¯2	3.744E¯2	8.584E¯3	8.581E¯3	2.292E¯1	2.292E¯1
2.604E¯2	2.603E¯2	5.967E¯3	5.965E¯3	2.292E¯1	2.292E¯1
1.807E¯2	1.807E¯2	4.142E¯3	4.142E¯3	2.292E¯1	2.292E¯1
1.253E¯2	1.253E¯2	2.872E¯3	2.872E¯3	2.292E¯1	2.292E¯1

```
      SIGDIG 6
4
```

(5.5)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],(-0.5×Y[1])-1.5×Y[2]
    ∇

      X
0  0.2  2  4
      Y
1  ¯1

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.18733E¯1	¯1.63747E¯1	2.58026E¯6	5.16052E¯7
4.00000E¯1	6.70324E¯1	¯1.34065E¯1	4.22509E¯6	8.45017E¯7
6.00000E¯1	5.48817E¯1	¯1.09763E¯1	5.18882E¯6	1.03776E¯6
8.00000E¯1	4.49335E¯1	¯8.98669E¯2	5.66434E¯6	1.13287E¯6
1.00000E0	3.67885E¯1	¯7.35770E¯2	5.79697E¯6	1.15939E¯6
1.20000E0	3.01200E¯1	¯6.02400E¯2	5.69540E¯6	1.13908E¯6
1.40000E0	2.46602E¯1	¯4.93205E¯2	5.44017E¯6	1.08803E¯6
1.60000E0	2.01902E¯1	¯4.03803E¯2	5.09034E¯6	1.01807E¯6
1.80000E0	1.65304E¯1	¯3.30607E¯2	4.68857E¯6	9.37715E¯7
2.00000E0	1.35340E¯1	¯2.70679E¯2	4.26521E¯6	8.53041E¯7

```
      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←(-0.5×Y[1])-1.5×Y[2]
    ∇

      USE2 K2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.18737E¯1	¯1.63748E¯1	5.94026E¯6	1.60344E¯6
4.00000E¯1	6.70329E¯1	¯1.34067E¯1	9.36932E¯6	2.58982E¯6
6.00000E¯1	5.48823E¯1	¯1.09765E¯1	1.10364E¯5	3.13356E¯6
8.00000E¯1	4.49340E¯1	¯8.98692E¯2	1.14985E¯5	3.36581E¯6
1.00000E0	3.67891E¯1	¯7.35793E¯2	1.11654E¯5	3.38440E¯6
1.20000E0	3.01205E¯1	¯6.02421E¯2	1.03355E¯5	3.26168E¯6
1.40000E0	2.46606E¯1	¯4.93224E¯2	9.22270E¯6	3.05056E¯6
1.60000E0	2.01904E¯1	¯4.03821E¯2	7.97741E¯6	2.78918E¯6
1.80000E0	1.65306E¯1	¯3.30623E¯2	6.70293E¯6	2.50456E¯6
2.00000E0	1.35341E¯1	¯2.70693E¯2	5.46765E¯6	2.21541E¯6

```
      S1←SAVE
```

(5.5)

```
      USE2 R2
      X RK2 Y

      X              Y              V              |E(Y)|         |E(V)|

  2.00000E¯1     8.18737E¯1     ¯1.63748E¯1    5.94026E¯6     1.44117E¯6
  4.00000E¯1     6.70330E¯1     ¯1.34066E¯1    9.50904E¯6     2.33808E¯6
  6.00000E¯1     5.48823E¯1     ¯1.09765E¯1    1.13916E¯5     2.84276E¯6
  8.00000E¯1     4.49341E¯1     ¯8.98689E¯2    1.21009E¯5     3.06981E¯6
  1.00000E0      3.67891E¯1     ¯7.35790E¯2    1.20172E¯5     3.10500E¯6

  1.20000E0      3.01206E¯1     ¯6.02419E¯2    1.14202E¯5     3.01195E¯6
  1.40000E0      2.46607E¯1     ¯4.93222E¯2    1.05125E¯5     2.83740E¯6
  1.60000E0      2.01906E¯1     ¯4.03819E¯2    9.43909E¯6     2.61520E¯6
  1.80000E0      1.65307E¯1     ¯3.30621E¯2    8.30116E¯6     2.36950E¯6
  2.00000E0      1.35342E¯1     ¯2.70692E¯2    7.16769E¯6     2.11716E¯6

      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(*-X),-*-X
    ∇

      RATIO X

                        RATIO OF ERRORS

      E(R1)/E(K2)               E(R1)/E(R2)               E(K2)/E(R2)
     Y           V             Y           V             Y           V

    0.4344      0.3218      0.4344      0.3581      1           1.113
    0.4509      0.3263      0.4443      0.3614      0.9853      1.108
    0.4702      0.3312      0.4555      0.3651      0.9688      1.102
    0.4926      0.3366      0.4681      0.369       0.9502      1.096
    0.5192      0.3426      0.4824      0.3734      0.9291      1.09

    0.5511      0.3492      0.4987      0.3782      0.905       1.083
    0.5899      0.3567      0.5175      0.3835      0.8773      1.075
    0.6381      0.365       0.5393      0.3893      0.8451      1.067
    0.6995      0.3744      0.5648      0.3957      0.8075      1.057
    0.7801      0.385       0.5951      0.4029      0.7628      1.046

      SIGDIG 6
4
```

(5.6)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],(11×Y[1])+10×Y[2]
    ∇

      X
0  0.2  2  4
      Y
1  ¯1

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.18733E¯1	¯1.63747E¯1	2.58026E¯6	5.16052E¯7
4.00000E¯1	6.70324E¯1	¯1.34065E¯1	4.22509E¯6	8.45017E¯7
6.00000E¯1	5.48817E¯1	¯1.09763E¯1	5.18882E¯6	1.03776E¯6
8.00000E¯1	4.49335E¯1	¯8.98669E¯2	5.66434E¯6	1.13287E¯6
1.00000E0	3.67885E¯1	¯7.35770E¯2	5.79697E¯6	1.15939E¯6
1.20000E0	3.01200E¯1	¯6.02400E¯2	5.69540E¯6	1.13908E¯6
1.40000E0	2.46602E¯1	¯4.93205E¯2	5.44017E¯6	1.08804E¯6
1.60000E0	2.01902E¯1	¯4.03803E¯2	5.09033E¯6	1.01807E¯6
1.80000E0	1.65304E¯1	¯3.30607E¯2	4.68855E¯6	9.37757E¯7
2.00000E0	1.35340E¯1	¯2.70679E¯2	4.26504E¯6	8.53399E¯7

```
      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←(11×Y[1])+10×Y[2]
    ∇

      USE2 K2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	8.19010E¯1	¯1.63140E¯1	2.79487E¯4	6.06065E¯4
4.00000E¯1	6.72882E¯1	¯1.28432E¯1	2.56161E¯3	5.63219E¯3
6.00000E¯1	5.70626E¯1	¯6.16772E¯2	2.18142E¯2	4.80851E¯2
8.00000E¯1	6.34042E¯1	3.17450E¯1	1.84713E¯1	4.07316E¯1
1.00000E0	1.93125E0	3.37404E0	1.56337E0	3.44762E0
1.20000E0	1.35329E1	2.91190E1	1.32317E1	2.91792E1
1.40000E0	1.12233E2	2.46910E2	1.11987E2	2.46959E2
1.60000E0	9.48006E2	2.09011E3	9.47804E2	2.09015E3
1.80000E0	8.02193E3	1.76900E4	8.02177E3	1.76900E4
2.00000E0	6.78926E4	1.49720E5	6.78925E4	1.49720E5

```
      S1←SAVE
```

(5.6)

```
      USE2 R2
      X RK2 Y

      X               Y               V               |E(Y)|          |E(V)|

  2.00000E¯1      8.19010E¯1     ¯1.63131E¯1      2.79487E¯4      6.15040E¯4
  4.00000E¯1      6.72910E¯1     ¯1.28364E¯1      2.59014E¯3      5.69966E¯3
  6.00000E¯1      5.70882E¯1     ¯6.11969E¯2      2.20701E¯2      4.85655E¯2
  8.00000E¯1      6.35941E¯1      3.20774E¯1      1.86612E¯1      4.10640E¯1
  1.00000E0       1.94458E0       3.39596E0       1.57670E0       3.46954E0

  1.20000E0       1.36220E1       2.92522E1       1.33208E1       2.93124E1
  1.40000E0       1.12786E2       2.47595E2       1.12540E2       2.47644E2
  1.60000E0       9.50987E2       2.09217E3       9.50785E2       2.09221E3
  1.80000E0       8.03281E3       1.76758E4       8.03264E3       1.76759E4
  2.00000E0       6.78634E4       1.49333E5       6.78633E4       1.49334E5

      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(*-X),-*-X
    ∇

      RATIO X

                              RATIO OF ERRORS

      E(R1)/E(K2)                E(R1)/E(R2)                E(K2)/E(R2)
     Y            V            Y            V            Y            V

  9.232E¯3     8.515E¯4     9.232E¯3     8.391E¯4     1.000E0      9.854E¯1
  1.649E¯3     1.500E¯4     1.631E¯3     1.483E¯4     9.890E¯1     9.882E¯1
  2.379E¯4     2.158E¯5     2.351E¯4     2.137E¯5     9.884E¯1     9.901E¯1
  3.067E¯5     2.781E¯6     3.035E¯5     2.759E¯6     9.898E¯1     9.919E¯1
  3.708E¯6     3.363E¯7     3.677E¯6     3.342E¯7     9.915E¯1     9.937E¯1

  4.304E¯7     3.904E¯8     4.276E¯7     3.886E¯8     9.933E¯1     9.955E¯1
  4.858E¯8     4.406E¯9     4.834E¯8     4.394E¯9     9.951E¯1     9.972E¯1
  5.371E¯9     4.871E¯10    5.354E¯9     4.866E¯10    9.969E¯1     9.990E¯1
  5.845E¯10    5.301E¯11    5.837E¯10    5.305E¯11    9.986E¯1     1.001E0
  6.282E¯11    5.700E¯12    6.285E¯11    5.715E¯12    1.000E0      1.003E0

      SIGDIG 6
4
```

(5.7)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],Y[1]
    ∇

      X
0  0.2  2  4
      Y
2  0

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	2.04013E0	8.05333E¯2	1.77914E¯7	1.06769E¯6
4.00000E¯1	2.16214E0	1.64299E¯1	2.51259E¯6	2.19255E¯6
6.00000E¯1	2.37092E0	2.54658E¯1	7.15530E¯6	3.50659E¯6
8.00000E¯1	2.67486E0	3.55237E¯1	1.44385E¯5	5.15343E¯6
1.00000E0	3.08614E0	4.70073E¯1	2.48951E¯5	7.29780E¯6
1.20000E0	3.62127E0	6.03774E¯1	3.92894E¯5	1.01360E¯5
1.40000E0	4.30174E0	7.61707E¯1	5.86617E¯5	1.39084E¯5
1.60000E0	5.15484E0	9.50208E¯1	8.43886E¯5	1.89139E¯5
1.80000E0	6.21483E0	1.17684E0	1.18262E¯4	2.55279E¯5
2.00000E0	7.52423E0	1.45071E0	1.62593E¯4	3.42247E¯5

```
      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←Y[1]
    ∇

      USE2 B2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	2.04013E0	8.05344E¯2	1.27040E¯10	3.05327E¯10
4.00000E¯1	2.16214E0	1.64301E¯1	1.58977E¯9	3.49658E¯9
6.00000E¯1	2.37093E0	2.54661E¯1	7.36648E¯9	9.81309E¯9
8.00000E¯1	2.67487E0	3.55242E¯1	2.07791E¯8	1.97394E¯8
1.00000E0	3.08616E0	4.70080E¯1	4.57508E¯8	3.40340E¯8
1.20000E0	3.62131E0	6.03784E¯1	8.70988E¯8	5.37742E¯8
1.40000E0	4.30180E0	7.61721E¯1	1.50881E¯7	8.04195E¯8
1.60000E0	5.15493E0	9.50227E¯1	2.44817E¯7	1.15897E¯7
1.80000E0	6.21495E0	1.17687E0	3.78802E¯7	1.62714E¯7
2.00000E0	7.52439E0	1.45074E0	5.65552E¯7	2.24101E¯7

```
      S1←SAVE
```

(5.7)

```
      USE2 R2
      X RK2 Y

      X              Y              V              |E(Y)|         |E(V)|

  2.00000E¯1     2.04013E0      8.05344E¯2     3.54219E¯8     2.70349E¯8
  4.00000E¯1     2.16214E0      1.64301E¯1     9.84530E¯8     5.05378E¯8
  6.00000E¯1     2.37093E0      2.54662E¯1     1.87501E¯7     7.24139E¯8
  8.00000E¯1     2.67487E0      3.55242E¯1     3.02902E¯7     9.43562E¯8
  1.00000E0      3.08616E0      4.70081E¯1     4.46807E¯7     1.17946E¯7

  1.20000E0      3.62131E0      6.03785E¯1     6.23166E¯7     1.44747E¯7
  1.40000E0      4.30180E0      7.61721E¯1     8.37815E¯7     1.76387E¯7
  1.60000E0      5.15493E0      9.50227E¯1     1.09866E¯6     2.14653E¯7
  1.80000E0      6.21495E0      1.17687E0      1.41596E¯6     2.61576E¯7
  2.00000E0      7.52439E0      1.45074E0      1.80275E¯6     3.19540E¯7

      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←((*X)+*-X),(*X)-*-X
    ∇

      RATIO X

                          RATIO OF ERRORS

      E(R1)/E(B2)              E(R1)/E(R2)              E(B2)/E(R2)
     Y           V            Y           V            Y           V

  1.400E3     3.497E3      5.023E0     3.949E1      3.586E¯3    1.129E¯2
  1.580E3     6.271E2      2.552E1     4.338E1      1.615E¯2    6.919E¯2
  9.713E2     3.573E2      3.816E1     4.842E1      3.929E¯2    1.355E¯1
  6.949E2     2.611E2      4.767E1     5.462E1      6.860E¯2    2.092E¯1
  5.441E2     2.144E2      5.572E1     6.187E1      1.024E¯1    2.886E¯1

  4.511E2     1.885E2      6.305E1     7.003E1      1.398E¯1    3.715E¯1
  3.888E2     1.729E2      7.002E1     7.885E1      1.801E¯1    4.559E¯1
  3.447E2     1.632E2      7.681E1     8.811E1      2.228E¯1    5.399E¯1
  3.122E2     1.569E2      8.352E1     9.759E1      2.675E¯1    6.221E¯1
  2.875E2     1.527E2      9.019E1     1.071E2      3.137E¯1    7.013E¯1

      SIGDIG 6
4
```

(5.8)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],Y[1]-2×SIN X
    ∇

      X
0  0.2  2  4
      Y
1  0

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.01740E0	3.22667E¯2	4.27987E¯6	5.10186E¯7
4.00000E¯1	1.05975E0	5.01475E¯2	7.62708E¯6	6.68461E¯7
6.00000E¯1	1.11346E0	5.53043E¯2	1.03957E¯5	5.26346E¯7
8.00000E¯1	1.16670E0	4.94754E¯2	1.28837E¯5	1.22739E¯7
1.00000E0	1.20937E0	3.44851E¯2	1.53500E¯5	5.13818E¯7
1.20000E0	1.23325E0	1.22341E¯2	1.80286E¯5	1.36379E¯6
1.40000E0	1.23207E0	¯1.53235E¯2	2.11405E¯5	2.41599E¯6
1.60000E0	1.20150E0	¯4.62155E¯2	2.49053E¯5	3.66757E¯6
1.80000E0	1.13918E0	¯7.84951E¯2	2.95515E¯5	5.12461E¯6
2.00000E0	1.04467E0	¯1.10290E¯1	3.53272E¯5	6.80322E¯6

```
      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←Y[1]-2×SIN X
    ∇

      USE2 B2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	1.01740E0	3.22672E¯2	3.89826E¯7	8.17177E¯9
4.00000E¯1	1.05974E0	5.01482E¯2	7.66000E¯7	4.36508E¯9
6.00000E¯1	1.11345E0	5.53048E¯2	1.13392E¯6	3.71392E¯8
8.00000E¯1	1.16668E0	4.94755E¯2	1.49978E¯6	8.97733E¯8
1.00000E0	1.20935E0	3.44844E¯2	1.87124E¯6	1.62039E¯7
1.20000E0	1.23323E0	1.22325E¯2	2.25807E¯6	2.53894E¯7
1.40000E0	1.23204E0	¯1.53263E¯2	2.67281E¯6	3.65620E¯7
1.60000E0	1.20147E0	¯4.62197E¯2	3.13143E¯6	4.97945E¯7
1.80000E0	1.13914E0	¯7.85008E¯2	3.65401E¯6	6.52219E¯7
2.00000E0	1.04463E0	¯1.10297E¯1	4.26556E¯6	8.30622E¯7

```
      S1←SAVE
```

(5.8)

```
      USE2 R2
      X RK2 Y

      X                Y                V                |E(Y)|           |E(V)|

 2.00000E¯1      1.01740E0        3.22672E¯2       9.41106E¯7       3.32008E¯8
 4.00000E¯1      1.05974E0        5.01482E¯2       1.83438E¯6       1.71128E¯8
 6.00000E¯1      1.11345E0        5.53047E¯2       2.69095E¯6       4.63010E¯8
 8.00000E¯1      1.16668E0        4.94754E¯2       3.52328E¯6       1.55382E¯7
 1.00000E0       1.20935E0        3.44843E¯2       4.34679E¯6       3.08838E¯7

 1.20000E0       1.23323E0        1.22322E¯2       5.18147E¯6       5.05830E¯7
 1.40000E0       1.23204E0       ¯1.53267E¯2       6.05336E¯6       7.46217E¯7
 1.60000E0       1.20146E0       ¯4.62202E¯2       6.99611E¯6       1.03082E¯6
 1.80000E0       1.13914E0       ¯7.85016E¯2       8.05245E¯6       1.36179E¯6
 2.00000E0       1.04462E0       ¯1.10298E¯1       9.27585E¯6       1.74302E¯6

      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←((*-X)+SIN X),(-*-X)+COS X
    ∇

      RATIO X

                              RATIO OF ERRORS

      E(R1)/E(B2)                E(R1)/E(R2)                E(B2)/E(R2)
     Y            V             Y            V             Y            V

   10.98        62.43         4.548        15.37         0.4142       0.2461
    9.957      153.1          4.158        39.06         0.4176       0.2551
    9.168       14.17         3.863        11.37         0.4214       0.8021
    8.59         1.367        3.657         0.7899       0.4257       0.5778
    8.203        3.171        3.531         1.664        0.4305       0.5247

    7.984        5.371        3.479         2.696        0.4358       0.5019
    7.909        6.608        3.492         3.238        0.4415       0.49
    7.953        7.365        3.56          3.558        0.4476       0.4831
    8.087        7.857        3.67          3.763        0.4538       0.4789
    8.282        8.191        3.809         3.903        0.4599       0.4765

      SIGDIG 6
4
```

(5.9)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],-Y[1]
    ∇

      X
0  0.2  2  4
      Y
0  1

      USE1 R1
      X RK1 Y

      X                Y                V             |E(Y)|           |E(V)|

   2.00000E¯1     1.98667E¯1     1.96013E¯1     2.66412E¯6     1.77653E¯8
   4.00000E¯1     3.89413E¯1     1.84212E¯1     5.18673E¯6     2.46532E¯7
   6.00000E¯1     5.64635E¯1     1.65068E¯1     7.25767E¯6     6.71558E¯7
   8.00000E¯1     7.17347E¯1     1.39343E¯1     8.59456E¯6     1.26206E¯6
   1.00000E0      8.41462E¯1     1.08062E¯1     8.96198E¯6     1.97300E¯6

   1.20000E0      9.32031E¯1     7.24743E¯2     8.18818E¯6     2.74771E¯6
   1.40000E0      9.85444E¯1     3.39969E¯2     6.17815E¯6     3.52133E¯6
   1.60000E0      9.99571E¯1    ¯5.83568E¯3     2.92242E¯6     4.22471E¯6
   1.80000E0      9.73849E¯1    ¯4.54356E¯2     1.49896E¯6     4.78869E¯6
   2.00000E0      9.09304E¯1    ¯8.32242E¯2     6.91764E¯6     5.14853E¯6

      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←-Y[1]
    ∇

      USE2 B2
      X RK2 Y

      X                Y                V             |E(Y)|           |E(V)|

   2.00000E¯1     1.98669E¯1     1.96013E¯1     2.53827E¯9     7.09842E¯9
   4.00000E¯1     3.89418E¯1     1.84212E¯1     1.20140E¯8     1.37828E¯8
   6.00000E¯1     5.64643E¯1     1.65067E¯1     2.77805E¯8     1.95095E¯8
   8.00000E¯1     7.17356E¯1     1.39341E¯1     4.86638E¯8     2.37929E¯8
   1.00000E0      8.41471E¯1     1.08060E¯1     7.30480E¯8     2.62184E¯8

   1.20000E0      9.32039E¯1     7.24716E¯2     9.89536E¯8     2.65004E¯8
   1.40000E0      9.85450E¯1     3.39935E¯2     1.24166E¯7     2.44707E¯8
   1.60000E0      9.99574E¯1    ¯5.83988E¯3     1.46368E¯7     2.01060E¯8
   1.80000E0      9.73848E¯1    ¯4.54404E¯2     1.63285E¯7     1.35301E¯8
   2.00000E0      9.09298E¯1    ¯8.32294E¯2     1.72831E¯7     5.01159E¯9

      S1←SAVE
```

(5.9)

```
      USE2 R2
      X RK2 Y

      X              Y              V              |E(Y)|         |E(V)|

  2.00000E¯1     1.98669E¯1     1.96013E¯1     2.54994E¯9     1.49827E¯8
  4.00000E¯1     3.89418E¯1     1.84212E¯1     1.63372E¯8     2.64235E¯8
  6.00000E¯1     5.64643E¯1     1.65067E¯1     3.76628E¯8     3.34745E¯8
  8.00000E¯1     7.17356E¯1     1.39341E¯1     6.21940E¯8     3.55991E¯8
  1.00000E0      8.41471E¯1     1.08060E¯1     8.52927E¯8     3.25873E¯8

  1.20000E0      9.32039E¯1     7.24716E¯2     1.02330E¯7     2.46010E¯8
  1.40000E0      9.85450E¯1     3.39934E¯2     1.09024E¯7     1.21349E¯8
  1.60000E0      9.99574E¯1    ¯5.83991E¯3     1.01759E¯7     3.99562E¯9
  1.80000E0      9.73848E¯1    ¯4.54404E¯2     7.78560E¯8     2.27021E¯8
  2.00000E0      9.09297E¯1    ¯8.32294E¯2     3.58011E¯8     4.26844E¯8

      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(SIN X),COS X
    ∇

      RATIO X

                              RATIO OF ERRORS

      E(R1)/E(B2)              E(R1)/E(R2)              E(B2)/E(R2)
     Y           V            Y           V            Y           V

 1050          2.503    1045          1.186        0.9954      0.4738
  431.7       17.89      317.5        9.33         0.7354      0.5216
  261.3       34.42      192.7       20.06         0.7376      0.5828
  176.6       53.04      138.2       35.45         0.7825      0.6684
  122.7       75.25      105.1       60.54         0.8564      0.8046

   82.75     103.7        80.02     111.7          0.967       1.077
   49.76     143.9        56.67     290.2          1.139       2.017
   19.97     210.1        28.72    1057            1.438       5.032
    9.18     353.9        19.25     210.9          2.097       0.596
   40.03    1027         193.2      120.6          4.828       0.1174

      SIGDIG 6
4
```

(5.10)

```
      ∇Y1PRIME[⎕]∇
    ∇ V←X Y1PRIME Y
[1]   V←Y[2],-Y[1]
    ∇

      X
0  0.2  2  4
      Y
1  0

      USE1 R1
      X RK1 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	9.80067E¯1	¯3.97333E¯2	8.88264E¯8	5.32823E¯7
4.00000E¯1	9.21062E¯1	¯7.78826E¯2	1.23266E¯6	1.03735E¯6
6.00000E¯1	8.25339E¯1	¯1.12927E¯1	3.35779E¯6	1.45153E¯6
8.00000E¯1	6.96713E¯1	¯1.43469E¯1	6.31032E¯6	1.71891E¯6
1.00000E0	5.40312E¯1	¯1.68292E¯1	9.86498E¯6	1.79240E¯6
1.20000E0	3.62371E¯1	¯1.86406E¯1	1.37386E¯5	1.63764E¯6
1.40000E0	1.69985E¯1	¯1.97089E¯1	1.76067E¯5	1.23563E¯6
1.60000E0	¯2.91784E¯2	¯1.99914E¯1	2.11235E¯5	5.84483E¯7
1.80000E0	¯2.27178E¯1	¯1.94770E¯1	2.39435E¯5	2.99792E¯7
2.00000E0	¯4.16121E¯1	¯1.81861E¯1	2.57427E¯5	1.38353E¯6

```
      S←SAVE

      ∇Y2PRIME[⎕]∇
    ∇ V←X Y2PRIME Y
[1]   V←-Y[1]
    ∇

      USE2 B2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	9.80067E¯1	¯3.97339E¯2	6.34639E¯11	1.52099E¯10
4.00000E¯1	9.21061E¯1	¯7.78837E¯2	7.79762E¯10	1.70586E¯9
6.00000E¯1	8.25336E¯1	¯1.12928E¯1	3.50562E¯9	4.54557E¯9
8.00000E¯1	6.96707E¯1	¯1.43471E¯1	9.41942E¯9	8.44892E¯9
1.00000E0	5.40302E¯1	¯1.68294E¯1	1.95048E¯8	1.31036E¯8
1.20000E0	3.62358E¯1	¯1.86408E¯1	3.43037E¯8	1.81227E¯8
1.40000E0	1.69967E¯1	¯1.97090E¯1	5.40108E¯8	2.30695E¯8
1.60000E0	¯2.91996E¯2	¯1.99915E¯1	7.83624E¯8	2.74846E¯8
1.80000E0	¯2.27202E¯1	¯1.94770E¯1	1.06637E¯7	3.09140E¯8
2.00000E0	¯4.16147E¯1	¯1.81860E¯1	1.37678E¯7	3.29389E¯8

```
      S1←SAVE
```

(5.10)

```
      USE2 R2
      X RK2 Y
```

X	Y	V	|E(Y)|	|E(V)|
2.00000E¯1	9.80067E¯1	¯3.97339E¯2	1.78390E¯8	1.43115E¯8
4.00000E¯1	9.21061E¯1	¯7.78837E¯2	4.96897E¯8	3.03202E¯8
6.00000E¯1	8.25336E¯1	¯1.12929E¯1	9.62414E¯8	4.67580E¯8
8.00000E¯1	6.96707E¯1	¯1.43471E¯1	1.56916E¯7	6.22732E¯8
1.00000E0	5.40302E¯1	¯1.68294E¯1	2.29920E¯7	7.55152E¯8
1.20000E0	3.62357E¯1	¯1.86408E¯1	3.12135E¯7	8.52143E¯8
1.40000E0	1.69967E¯1	¯1.97090E¯1	3.99401E¯7	9.02637E¯8
1.60000E0	¯2.92000E¯2	¯1.99915E¯1	4.86648E¯7	8.97918E¯8
1.80000E0	¯2.27203E¯1	¯1.94770E¯1	5.68169E¯7	8.32240E¯8
2.00000E0	¯4.16147E¯1	¯1.81860E¯1	6.37944E¯7	7.03288E¯8

```
      S2←SAVE

      SIGDIG 4
6

      ∇YEXACT[⎕]∇
    ∇ M←D YEXACT X
[1]   M←Dρ0
[2]   M←(COS X),-SIN X
    ∇

      RATIO X
```

RATIO OF ERRORS

E(R1)/E(B2)		E(R1)/E(R2)		E(B2)/E(R2)	
Y	V	Y	V	Y	V
1.400E3	3.503E3	4.979E0	3.723E1	3.558E¯3	1.063E¯2
1.581E3	6.081E2	2.481E1	3.421E1	1.569E¯2	5.626E¯2
9.578E2	3.193E2	3.489E1	3.104E1	3.643E¯2	9.721E¯2
6.699E2	2.034E2	4.021E1	2.760E1	6.003E¯2	1.357E¯1
5.058E2	1.368E2	4.291E1	2.374E1	8.483E¯2	1.735E¯1
4.005E2	9.036E1	4.401E1	1.922E1	1.099E¯1	2.127E¯1
3.260E2	5.356E1	4.408E1	1.369E1	1.352E¯1	2.556E¯1
2.696E2	2.127E1	4.341E1	6.509E0	1.610E¯1	3.061E¯1
2.245E2	9.698E0	4.214E1	3.602E0	1.877E¯1	3.715E¯1
1.870E2	4.200E1	4.035E1	1.967E1	2.158E¯1	4.684E¯1

```
      SIGDIG 6
4
```